Appendix A Completion and the Cabalar Semantics

The following definitions are from (Bartholomew and Lee 2013).

We say that a formula $F$ is in Clark normal form (relative to a list $c$ of intensional constants) if it is a conjunction of sentences of the form

$$\forall x(G \rightarrow p(x))$$  \hspace{1cm} (A1)

and

$$\forall xy(G \rightarrow f(x) = y)$$  \hspace{1cm} (A2)

one for each intensional predicate $p$ and each intensional function $f$, where $x$ is a list of distinct object variables, $y$ is an object variable, and $G$ is an arbitrary formula that has no free variables other than those in $x$ and $y$.

The completion of a formula $F$ in Clark normal form (relative to $c$) is obtained from $F$ by replacing each conjunctive term (A1) with

$$\forall x(p(x) \leftrightarrow G)$$

and each conjunctive term (A2) with

$$\forall xy(f(x) = y \leftrightarrow G).$$

An occurrence of a symbol or a subformula in a formula $F$ is called strictly positive in $F$ if that occurrence is not in the antecedent of any implication in $F$. The dependency graph of $F$ (relative to $c$) is the directed graph that

- has all members of $c$ as its vertices, and
- has an edge from $c$ to $d$ if, for some strictly positive occurrence of $G \rightarrow H$ in $F$,
  - $c$ has a strictly positive occurrence in $H$, and
  - $d$ has a strictly positive occurrence in $G$.

We say that $F$ is tight (on $c$) if the dependency graph of $F$ (relative to $c$) is acyclic.

The following theorem relates the Cabalar semantics to completion, which follows immediately from Theorem 12 from (Bartholomew and Lee 2013) and Theorem 6.
Theorem 11
For any sentence \( F \) in Clark normal form that is tight on \( c \) and any total interpretation \( I \), if \( I \models \exists xy(x \neq y) \), then \( I \models^{CBL} [F; e] \) iff \( I \models^{SM} [F; e] \) iff \( I \) is a model of the completion of \( F \) relative to \( c \).

Appendix B  Review of the Balduccini Semantics

The following is a review of the Balduccini semantics. Let us restrict a signature \( \sigma \) to be comprised of a set of intensional function and predicate constants denoted \( c \) as well as a set of non-intensional object constants \( \sigma \setminus c \).

Balduccini considered terms to have the form \( f(c_1, \ldots, c_k) \) where \( f \) is an intensional function constant (in \( \sigma \)), and each \( c_i \) is a non-intensional object constant (in \( \sigma \setminus c \)). He considered an atom to be an expression \( p(c_1, \ldots, c_k) \) where \( p \) is an intensional predicate constant, and each \( c_i \) is a non-intensional object constant; a t-atom is an expression of the form \( f = g \) where \( f \) is a term and \( g \) is either a term or a non-intensional object constant; a seed t-atom is a t-atom of the form \( f = c \) where \( c \) is a non-intensional object constant. A t-literal is a t-atom \( f = g \) or \( \neg(f = g) \), where \( \neg \) denotes strong negation. A seed literal is an atom \( a \), or \( \neg a \), or a seed t-atom. A literal is an atom \( a \), or \( \neg a \), or a t-literal. An ASP\{f\} program consists of rules of the form

\[
h \leftarrow l_1, \ldots, l_m, \lnot l_{m+1}, \ldots, \lnot l_n ,
\]

where \( h \) is a seed literal or \( \bot \), and each \( l_i \) is a literal. An ASP\{f\} program is a finite set of rules. We identify rule (B1) with an implication

\[
l_1 \land \cdots \land l_m \land \lnot l_{m+1} \land \cdots \land \lnot l_n \rightarrow h ,
\]

and an ASP\{f\} program as the conjunction of all rules in it. Note that ASP\{f\} programs do not contain variables, and can be viewed as a special case of head-e-plain formulas.

A set \( I \) of seed literals is said to be consistent if it contains no pair of an atom \( a \) and its strong negation \( \neg a \); and contains no pair of seed t-atoms \( l = c_1 \) and \( l = c_2 \) such that \( c_1 \neq c_2 \). It is clear that any subset of a consistent set of seed literals is consistent as well.

The notion of satisfaction between a consistent set \( I \) of seed literals and literals, denoted by \( I \models^c \), is defined as follows.

- For a seed literal \( l \), \( I \models^c l \) if \( l \in I \);
- For a non-seed literal \( f = g \), \( I \models^c f = g \) if \( I \) contains both \( f = c \) and \( g = c \) for some object constant \( c \);
- For a non-seed literal \( \neg(f = g) \), \( I \models^c \neg(f = g) \) if \( I \) contains both \( f = c_1 \) and \( g = c_2 \) for some object constants \( c_1 \) and \( c_2 \) such that \( c_1 \neq c_2 \).

This notion of satisfaction is extended to formulas allowing \( \land, \neg \) and \( \leftarrow \) as in classical logic.

The reduct of an ASP\{f\} program \( \Pi \) relative to a consistent set \( I \) of seed literals is denoted \( \Pi^l \) and is defined as

\[
\Pi^l = \{ h \leftarrow l_1, \ldots, l_m \mid (B1) \in \Pi \text{ and } I \models \lnot l_{m+1} \land \cdots \land \lnot l_n \} .
\]

\( I \) is called a Balduccini answer set of \( \Pi \) if

- \( I \models^c \Pi^l \), and,
- for every proper subset \( J \) of \( I \), we have \( J \models^c \Pi^l \).
Appendix C Proofs

C.1 Proof of Theorem 1

We will often use the following notation. Let $\sigma$ be a first-order signature, let $c$ be a set of constants that is a subset of $\sigma$, and let $d$ be a set of constants not belonging to $\sigma$ and is similar to $c$.\(^1\) $J_d^c$ denotes the interpretation of signature $(\sigma \setminus c) \cup d$ obtained from $J$ by replacing every constant from $c$ with the corresponding constant from $d$. For two interpretations $I$ and $J$ of $\sigma$ that agree on all constants in $\sigma \setminus c$, we define $J_d^c \cup I$ to be the interpretation from the extended signature $\sigma \cup d$ such that

- $J_d^c \cup I$ agrees with $I$ on all constants in $c$;
- $J_d^c \cup I$ agrees with $J_d^c$ on all constants in $d$;
- $J_d^c \cup I$ agrees with both $I$ and $J$ on all constants in $\sigma \setminus c$.

**Lemma 1**

For any sentence $F$ of signature $\sigma$ and any interpretations $I$ and $J$ of $\sigma$,

(a) if $J_d^c \cup I \models F^*(d)$, then $I \models F$;
(b) if $(I, I) \models F$, then $(I, I) \models F$.

**Proof.** By induction on $F$. \(\blacksquare\)

**Lemma 2**

Let $F$ be a sentence of signature $\sigma$, and let $I$ and $J$ be interpretations of $\sigma$ such that $J <^e I$. We have $J_d^c \cup I \models F^*(d)$ iff $J \models gr_I[F]^L$.

**Proof.** By induction on $F$.

**Case 1:** $F$ is an atomic sentence. Then $F^*(d)$ is $F(d) \land F$, where $F(d)$ is obtained from $F$ by replacing the members of $e$ with the corresponding members of $d$. Consider the following subcases:

- **Subcase 1:** $I \notmodels F$. Then $J_d^c \cup I \notmodels F^*(d)$. Further, $gr_I[F]^L = \bot$, so $J \notmodels gr_I[F]^L$.
- **Subcase 2:** $I \models F$. Then $J_d^c \cup I \models F^*(d)$ iff $J_d^c \models F(d)$ iff $J \models F$. Further, $gr_I[F]^L = F$, so $J \models gr_I[F]^L$ iff $J \models F$.

**Case 2:** $F$ is $G \land H$ or $G \lor H$. The claim follows immediately from I.H. on $G$ and $H$.

**Case 3:** $F$ is $G \rightarrow H$. Then $F^*(d) = (G^*(d) \rightarrow H^*(d)) \land (G \rightarrow H)$. Consider the following subcases:

- **Subcase 1:** $I \notmodels G \rightarrow H$. Then $J_d^c \cup I \notmodels F^*(d)$. Further, $gr_I[F]^L = \bot$, which $J$ does not satisfy.
- **Subcase 2:** $I \models G \rightarrow H$. Then $J_d^c \cup I \models F^*(d)$ iff $J_d^c \cup I \models G^*(d) \rightarrow H^*(d)$. On the other hand, $gr_I[F]^L = gr_I[G]^L \rightarrow gr_I[H]^L$ so this case holds by I.H. on $G$ and $H$.

**Case 4:** $F$ is $\exists x G(x)$. By I.H., $J_d^c \cup I \models G(\xi)^*(d)$ iff $J \models gr_I[G(\xi)^]L$ for each $\xi \in \|I\|$. The claim follows immediately.

**Case 5:** $F$ is $\forall x G(x)$. Similar to Case 4. \(\blacksquare\)

---

\(^1\) That is to say, $d$ and $c$ have the same length and the corresponding members are either predicate constants of the same arity or function constants of the same arity.
Lemma 3
For any interpretations $I$ and $J$ of signature $\sigma$, we have $J^c_d \cup I \models d < c \iff J <^e I$.

Proof. Recall that by definition, $d < c$ is

$$(d^{pred} \leq c^{pred}) \land \neg(d = c),$$

and by definition, $J <^e I$ is

- $J$ and $I$ have the same universe and agree on all constants not in $c$;
- $p^J \subseteq p^I$ for all predicate constants $p$ in $c$; and
- $J$ and $I$ do not agree on $c$.

First, by the definition of $J^c_d \cup I$, $J$ and $I$ have the same universe and agree on all constants in $\sigma \setminus c$.

Second, by definition, $J^c_d \cup I \models d^{pred} \leq c^{pred}$ iff, for every predicate constant $p$ in $c$,

$$J^c_d \cup I \models \forall x (p(x)^J \rightarrow p(x)),$$

which is equivalent to saying that $(p^J)^c_d \cup I \subseteq p^{J^c_d \cup I}$. Since $I$ does not interpret any constant from $d$, and $J^c_d$ does not interpret any constant from $c$, this is equivalent to $(p^J)^c_d \subseteq p^I$ and further to $p^J \subseteq p^I$.

Third, since $I$ does not interpret any constant from $d$ and $J^c_d$ does not interpret any constant from $c$, $J^c_d \cup I \models \neg(d = c)$ is equivalent to saying that $J$ and $I$ do not agree on $c$.

Theorem 1 Let $F$ be a first-order sentence of signature $\sigma$ and $c$ be a list of intensional constants. For any interpretation $I$ of $\sigma$, $I \models SM[F; c]$ iff

- $I$ satisfies $F$, and
- every interpretation $J$ such that $J <^e I$ does not satisfy $(gr_I[F])^L$.

Proof. $I \models SM[F; c]$ is by definition

$$I \models F \land \neg \exists \hat{c}(\hat{c} < c \land F^+(\hat{c})).$$

(C1)

The first item, “$I$ satisfies $F$”, is equivalent to the first conjunctive term of (C1).

By Lemma 2 and Lemma 3, the second item, “no interpretation $J$ of $\sigma$ such that $J <^e I$ satisfies $gr_I[F]^L$”, is equivalent to the second conjunctive term in (C1).

C.2 Proofs of Theorem 2 and Theorem 3

Recall the definition: $J \preceq^e I$ if

- $J$ and $I$ have the same universe and agree on all constants not in $c$;
- $p^J \subseteq p^I$ for all predicate constants in $c$; and
- $f^I(\xi) = u$ or $f^J(\xi) = f^I(\xi)$ for all function constants in $c$ and all lists $\xi$ of elements in the universe.

As before, let $d$ be a list of constants that is similar to $c$ and is disjoint from $\sigma$. The notion of $J^c_d \cup I$ is straightforwardly extended to the case when $J$ and $I$ are partial interpretations.

$^2 p(x)^c_d$ denotes the atom that is obtained from $p(x)$ by replacing $p$ with the corresponding member of $d$ if $p \in c$, and no change otherwise.
Lemma 4
For any partial interpretations \( I \) and \( J \) of signature \( \sigma \), we have \( J \preceq^c I \) iff \( J^c_d \cup I \vDash_p d \preceq c \).

Proof. By the definition of \( J^c_d \cup I \), \( J \) and \( I \) have the same universe and agree on all constants in \( \sigma \setminus c \), which is the first condition of \( J \preceq^c I \).

Recall the definition: \( d \preceq c \) is

\[
(d^{\text{pred}} \leq c^{\text{pred}}) \land (d^{\text{func}} \leq c^{\text{func}}).
\]

\( J^c_d \cup I \vDash_p d^{\text{pred}} \leq c^{\text{pred}} \) iff, for every predicate constant \( p \) in \( c \),

\[
J^c_d \cup I \vDash_p \forall x(p(x)^c_d \rightarrow p(x)),
\]

which is equivalent to saying that \( (p)^c_d \cup I \preceq (p)^c_d \cup I \). Since \( I \) does not interpret any constant from \( d \) and \( J^c_d \) does not interpret any constant from \( c \), this is equivalent to \( (p)^c_d \cup I \preceq p^I \) and further to \( p^I \subseteq p^I \), which is the second condition of \( J \preceq^c I \).

\( J^c_d \cup I \vDash_p (d^{\text{func}} \leq c^{\text{func}}) \) iff, for every function constant \( f \) in \( c \),

\[
J^c_d \cup I \vDash_p \forall x((f(x)^c_d \neq f(x)^c_d) \lor (f(x)^c_d = f(x))),
\]

which is equivalent to saying that \( f^I(\xi) = u \) or \( f^I(\xi) = f^I(\xi) \) for all \( \xi \), the third condition of \( J \preceq^c I \).}

Lemma 5
For any partial interpretations \( I \) and \( J \) of signature \( \sigma \), we have \( J \prec^c I \) iff \( J^c_d \cup I \vDash_p d \prec c \).

Proof. Immediate from Lemma 4 since

- \( J \prec^c I \) iff \( J \preceq^c I \) and not \( J \preceq^c J \), and
- \( J^c_d \cup I \vDash_p d \prec c \) iff \( J^c_d \cup I \vDash_p d \preceq c \) and \( J^c_d \cup I \nvdash_p c \preceq d \).

Lemma 6
For any sentence \( F \) of signature \( \sigma \) and any partial interpretations \( I \) and \( J \) of \( \sigma \) such that \( J \preceq^c I \),

(a) if \( J^c_d \cup I \vDash_p F^1(d) \), then \( I \vDash_p F \);

(b) if \( \langle J, I \rangle \vDash_{\text{pht}} F \), then \( \langle I, I \rangle \vDash_{\text{pht}} F \).

Proof. Each of (a) and (b) can be proved by induction on \( F \).

We will show only the case when \( F \) is an atomic sentence. The other cases are straightforward:

Part (a): Let \( F \) be an atomic sentence. Assume \( J^c_d \cup I \vDash_p F^1(d) \), i.e., \( J \vDash_p F \).

- Subcase 1: \( F \) is of the form \( p(t) \). Since \( J \preceq^c I \), it follows that \( I \vDash_p F \).

- Subcase 2: \( F \) is of the form \( t_1 = t_2 \). Since \( J^c_d \cup I \vDash_p F(d) \), \( t_1^I = t_2^I \neq u \). From \( J \preceq^c I \), it follows that \( t_1^I = t_2^I \neq u \), i.e., \( J \vDash_p F \).

Part (b): Let \( F \) be an atomic sentence. Assume \( \langle J, I \rangle \vDash_{\text{pht}} F \), i.e., \( \langle J, I \rangle, h \vDash_{\text{pht}} F \).

- Subcase 1: \( F \) is of the form \( p(t) \). Since \( J \preceq^c I \), it follows that \( \langle J, I \rangle, t \vDash_{\text{pht}} F \).

- Subcase 2: \( F \) is of the form \( t_1 = t_2 \). Since \( \langle J, I \rangle, h \vDash_{\text{pht}} F \), \( t_1^I = t_2^I \neq u \). From \( J \preceq^c I \), it follows that \( t_1^I = t_2^I \neq u \), i.e., \( \langle J, I \rangle, t \vDash_{\text{pht}} F \).
Lemma 7
Let $F$ be a sentence of signature $\sigma$, and let $I$ and $J$ be partial interpretations of $\sigma$ such that $J \preceq^c I$. We have $J \models_p gr_I[F]^L$ iff $\langle J, I \rangle \models_{\text{fin}} F$.

**Proof.** By induction on $F$.

Case 1: $F$ is an atomic sentence. Clearly, $gr_I[F]$ is $F$.

- **Subcase 1:** $I \not\models_F F$. Then $gr_I[F]^L$ is $\bot$, and $J \not\models_F \bot$. Further, since $\langle I, I \rangle \models_{\text{fin}} F$, by Lemma 6 (b), it follows that $\langle J, I \rangle \not\models_{\text{fin}} F$.
- **Subcase 2:** $I \models_F F$. Then $gr_I[F]^L$ is $F$. It is clear that $J \models_F F$ iff $\langle J, I \rangle \models_{\text{fin}} F$.

Case 2: $F$ is $G \land H$ or $G \lor H$. The claim follows immediately from I.H. on $G$ and $H$.

Case 3: $F$ is $G \rightarrow H$. Consider the following subcases:

- **Subcase 1:** $I \not\models_G G \rightarrow H$. $gr_I[G \rightarrow H]^L$ is $\bot$, and $J \not\models_H \bot$. Further, $\langle I, I \rangle \not\models_p G \rightarrow H$. By Lemma 6 (b), $\langle J, I \rangle \not\models_p G \rightarrow H$.
- **Subcase 2:** $I \models_G G \rightarrow H$. $gr_I[G \rightarrow H]^L$ is equivalent to $gr_I[G]^L \rightarrow gr_I[H]^L$. Further, $\langle J, I \rangle \models_{\text{fin}} G \rightarrow H$ is equivalent to saying that $\langle J, I \rangle \not\models_G G$ or $\langle J, I \rangle \models_{\text{fin}} H$. Then the claim follows from I.H. on $G$ and $H$.

Case 4: $F$ is $\forall x G(x)$, or $\exists x G(x)$. By induction on $G(\xi^c)$ for each $\xi$ in the universe. ■

Theorem 2
Let $F$ be a first-order sentence of signature $\sigma$ and let $c$ be a list of intensional constants. For any partial interpretation $I$ of $\sigma$, $\langle I, I \rangle$ is a partial equilibrium model of $F$ iff

- $I \models_p F$, and
- for every partial interpretation $J$ of $\sigma$ such that $J \preceq^c I$, we have $J \not\models_p gr_I[F]^L$.

**Proof.** Clearly, $I \models_p F$ iff $\langle I, I \rangle \models_{\text{fin}} F$. By Lemma 7, for every partial interpretation $J$ of $\sigma$ such that $J \preceq^c I$, $J \not\models_p gr_I[F]^L$ iff $\langle J, I \rangle \models_{\text{fin}} F$. ■

Lemma 8
Let $F$ be a sentence of signature $\sigma$, and let $I$ and $J$ be partial interpretations of $\sigma$. We have $J \models G \cup I \models F^I(d)$ iff $\langle J, I \rangle \models_{\text{fin}} F$.

**Proof.** By induction on $F$.

Case 1: $F$ is an atomic sentence. $F^I(d)$ is $F(d)$. $J \models G \cup I \models F(d)$ iff $J \models F$ iff $\langle J, I \rangle \models_{\text{fin}} F$ iff $\langle J, I \rangle \models_{\text{fin}} F$.

Case 2: $F$ is $G \land H$ or $G \lor H$. Follows by I.H. on $G$ and $H$.

Case 3: $F$ is $G \rightarrow H$. Consider the following subcases:

- **Subcase 1:** $I \not\models_G G \rightarrow H$. Clearly, $J \models G \rightarrow H$ and $\langle J, I \rangle \not\models_{\text{fin}} G \rightarrow H$.
- **Subcase 2:** $I \models_G G \rightarrow H$. Then $J \models G \rightarrow H$. Further, $\langle J, I \rangle \models_{\text{fin}} G \rightarrow H$ is equivalent to saying that $\langle J, I \rangle \not\models_G G$ or $\langle J, I \rangle \not\models_{\text{fin}} H$. Then the claim follows from I.H. on $G$ and $H$. ■
Case 4: \( F \) is \( \forall x G(x) \), or \( \exists x G(x) \). By induction on \( G(\xi^x) \) for each \( \xi \) in the universe. 

Theorem 3 For any sentence \( F \), a PHT-interpretation \( \langle I, I \rangle \) is a partial equilibrium model of \( F \) relative to \( c \) iff \( I \models \text{CBL}[F; c] \).

Proof. By definition, \( \text{CBL}[F; c] \) is
\[
F \land \neg \exists \hat{c} (\hat{c} < c \land F^\downarrow(\hat{c})).
\]
Clearly, \( I \models F \) iff \( \langle I, I \rangle \models \text{CBL}[F; c] \). From Lemma 5 and Lemma 8, it follows that \( I \models \neg \exists \hat{c} (\hat{c} < c \land F^\downarrow(\hat{c})) \) iff there is no interpretation \( J \) of \( \sigma \) such that \( J <^c I \) and \( \langle J, I \rangle \models \text{CBL}[F; c] \).

C.3 Proof of Theorem 4

Lemma 9
Let \( F \) be a sentence of signature \( \sigma \) and let \( I \) and \( J \) be interpretations of \( \sigma \) such that \( J <^c I \). We have \( J \models \text{gr}_I[F] \) iff \( \langle J, I \rangle \models \text{CBL} \).

Proof. By induction on \( F \).

Case 1: \( F \) is an atomic sentence. \( \text{gr}_I[F] \) is \( F \).

Case 2: \( F \) is \( G \land H \) or \( G \lor H \). The claim follows immediately from I.H. on \( G \) and \( H \).

Case 3: \( F \) is \( G \to H \). Consider the following subcases:

Case 3: \( I \models G \to H \). Then \( \text{gr}_I[G \to H] \) is \( \perp \), which \( J \) does not satisfy. Further, since \( \langle J, I \rangle \models \text{gr}_I[F] \), \( \langle J, I \rangle \models \text{CBL} \).

Case 4: \( F \) is \( \forall x G(x) \), or \( \exists x G(x) \). By induction on \( G(\xi^x) \) for each \( \xi \) in the universe.

Theorem 4 Let \( F \) be a first-order sentence of signature \( \sigma \) and \( c \) be a list of predicate and function constants. For any interpretation \( I \) of \( \sigma \), \( I \models \text{SM}[F; c] \) iff

- \( \langle I, I \rangle \models \text{CBL} \), and
- for every interpretation \( J \) of \( \sigma \) such that \( J <^c I \), we have \( \langle J, I \rangle \models \text{CBL} \).

Proof. We use Theorem 1 to refer to the reduct-based reformulation and instead show

- \( I \) satisfies \( F \), and
- every interpretation \( J \) such that \( J <^c I \) does not satisfy \( \text{gr}_I[F] \) iff
\( \{ I, I\}_{\text{fin}} \), \( F \), and

for every interpretation \( J \) of \( \sigma \), such that \( J \not\preceq I \), we have \( \langle J, I\rangle_{\text{fin}} \not\models F \).

Clearly, \( I \models F \iff \langle I, I\rangle_{\text{fin}} \models F \). By Lemma 9, for every interpretation \( J \) such that \( J \not\preceq I \), we have \( J \not\models (\text{gr}_1[F])_L \iff \langle J, I\rangle_{\text{fin}} \not\models F \).

\[ \]

### C.4 Proof of Theorem 5

**Lemma 10**

Let \( F \) be a \( \sigma \)-plain sentence of signature \( \sigma \), let \( I, K \) be total interpretations of \( \sigma \), and let \( J \) be a partial interpretation of \( \sigma \) such that

- \( J \not\preceq I \) and \( K \not\preceq I \);
- \( p^J = p^K \) for every predicate constant;
- \( f^J(\xi) = u \iff f^K(\xi) \neq f^I(\xi) \) for every function constant \( f \) and every \( \xi \in |I|^n \) where \( n \) is the arity of \( f \).

We have \( K \models \text{gr}_1[F]_L \iff J \not\models \text{gr}_1[F]_L \).

**Proof.**

Case 1: \( F \) is an atomic sentence of the form \( p(t) \). Since \( F \) is \( \sigma \)-plain, \( t \) contains no constants from \( c \), and by the assumption \( J \not\preceq I \) and \( K \not\preceq I \), we have \( t^J = t^K = t^I \). Since \( J \) and \( K \) agree on \( p \), the claim holds.

Case 2: \( F \) is an atomic sentence of the form \( f(t) = t_1 \).

- **Subcase 1:** \( I \not\models f(t) = t_1 \). Then \( \text{gr}_1[F]_L \) is \( \bot \), so the claim holds.
- **Subcase 2:** \( I \models f(t) = t_1 \). Then \( \text{gr}_1[F]_L \) is \( f(t) = t_1 \). Further, from the assumption that \( F \) is \( \sigma \)-plain, \( t \) and \( t_1 \) contain no constants from \( c \), and by the assumptions that \( J \not\preceq I \), \( K \not\preceq I \) and that \( I \) is total, we have \( t^J = t^K = t^I \) and \( t^J = t^K = t^I \) and \( t_1^J = t_1^K = t_1^I \).

Either \( f(t)_J \neq u \) or \( f(t)_I = u \). In the first case, since \( J \not\preceq I \), we have \( f(t)_J = f(t)_I \). Also, by the assumption on \( K \), \( f(t)_K = f(t)_I \). Consequently, \( J \not\models f(t) = t_1 \) and \( K \models f(t) = t_1 \).

In the second case, \( J \not\models f(t) = t_1 \). Also, by the assumption on \( K \), \( f(t)_K \neq f(t)_I \).

The other cases are straightforward.

Recall the definitions: for two classical interpretations \( I, K \) of the same signature \( \sigma \) with the same universe and a list \( c \) of distinct predicate and function constants, we write \( K \not\preceq I \) if

- \( K \) and \( I \) agree on all constants in \( \sigma \setminus c \), \hspace{1cm} (C2)
- \( p^K \subseteq p^I \) for all predicates \( p \) in \( c \), and \hspace{1cm} (C3)
- \( K \) and \( I \) do not agree on \( c \). \hspace{1cm} (C4)

Similarly, for two partial interpretations \( J \) and \( I \) of the same signature \( \sigma \) over the same universe \( |I| \), and a set of constants \( c \), \( J \not\preceq I \) is equivalent to

- \( J \) and \( I \) agree on all constants in \( \sigma \setminus c \), \hspace{1cm} (C5)
- \( p^J \subseteq p^I \) for all predicates \( p \) in \( c \), and \hspace{1cm} (C6)
- \( J \) and \( I \) do not agree on \( c \). \hspace{1cm} (C7)
with the additional requirement that

\[
\text{for every function constant } f \in \mathcal{C}, \text{ and every } \xi \in |I|^n \text{ where } n \text{ is the arity of } f, \quad f^J(\xi) = f^K(\xi) \text{ or } f^J(\xi) = u. \tag{C8}
\]

If we drop (C7), this is equivalent to \( J \prec^e I \).

**Lemma 11**

Let \( F \) be a \( \mathcal{C} \)-plain sentence of signature \( \sigma \), and let \( I \) be total interpretation of \( \sigma \) that satisfies \( \exists xy(x \neq y) \). There is a partial interpretation \( J \) such that \( J \prec^e I \) and \( J \models gr_F \). It follows from the assumption that there is a total interpretation \( K \) such that \( K \prec^e I \) and \( K \models gr_F \).

**Proof.** Left-to-right: Let \( J \) be a partial interpretation such that \( J \prec^e I \) and \( J \models gr_F \). We construct the total interpretation \( K \) as follows. For each constant \( d \) not in \( \mathcal{C} \), \( d^K = d^J = d^I \). For each predicate constant \( p \) in \( \mathcal{C} \) and each \( \xi \in |I|^n \) where \( n \) is the arity of \( p \),

\[
p^K(\xi) = p^J(\xi),
\]

and, for each function constant \( f \) in \( \mathcal{C} \) and each \( \xi \in |I|^n \) where \( n \) is the arity of \( f \),

\[
f^K(\xi) = \begin{cases} f^J(\xi) & \text{if } f^J(\xi) \neq u; \\ m(f^J(\xi)) & \text{otherwise} \end{cases}
\]

where \( m \) is a mapping \( m : |I| \to |I| \) such that \( \forall x(m(x) \neq x) \) (note that such a mapping requires \( I \models \exists xy(x \neq y) \)).

We now show that \( K \prec^e I \). It is immediate from the assumption \( J \prec^e I \) and by definition that (C2) and (C3) hold. Consider the following cases.

- **Case 1:** For every function constant \( f \in \mathcal{C} \) and every \( \xi \in |I|^n \) where \( n \) is the arity of \( f \), \( f^J(\xi) = f^K(\xi) \) (note that since \( I \) is total, these cannot be \( u \)). From (C7), it follows that there is at least one predicate constant \( p \in \mathcal{C} \) such that \( p^J \subseteq p^K \). However, by the definition of \( K \), \( p^K \subseteq p^J \) and so (C4) holds.

- **Case 2:** There is some function constant \( f \in \mathcal{C} \) and some \( \xi \in |I|^n \) where \( n \) is the arity of \( f \) such that \( f^J(\xi) \neq f^K(\xi) \). From (C8), it follows that \( f^J(\xi) = u \) and thus by the definition of \( K \), \( f^K(\xi) = m(f^J(\xi)) \neq f^J(\xi) \) and so (C4) holds.

By Lemma 10, the fact \( K \models gr_F \) follows from the assumption \( J \models gr_F \).

Right-to-left: Let \( K \) be a total interpretation such that \( K \prec^e I \) and \( K \models gr_F \). We construct the partial interpretation \( J \) as follows. For each constant \( d \) not in \( \mathcal{C} \), \( d^K = d^J = d^I \). For each predicate constant \( p \) in \( \mathcal{C} \) and each \( \xi \in |I|^n \) where \( n \) is the arity of \( p \),

\[
p^K(\xi) = p^J(\xi),
\]

and, for each function constant \( f \) in \( \mathcal{C} \) and each \( \xi \in |I|^n \) where \( n \) is the arity of \( f \),

\[
f^K(\xi) = \begin{cases} f^J(\xi) & \text{if } f^K(\xi) = f^J(\xi); \\ u & \text{otherwise} \end{cases}
\]

We now show that \( J \prec^e I \). It is immediate from the assumption that \( K \prec^e I \) and by definition that (C5) and (C6) hold. Consider the following cases.

- **Case 1:** For every function constant \( f \in \mathcal{C} \) and every \( \xi \in |I|^n \) where \( n \) is the arity of \( f \), \( f^K(\xi) = f^J(\xi) \). By the definition of \( J \), \( f^J(\xi) = f^K(\xi) \) and so (C8) holds. Now since
(C4) holds, there is at least one predicate constant \( p \) such that \( p^K \subset p^I \). However, by the definition of \( J, p^J \subset p^I \) and so (C7) holds.

- Case 2: There is some function constant \( f \in c \) and some \( \xi \in |I|^n \) where \( n \) is the arity of \( f \) such that \( f^K(\xi) \neq f^I(\xi) \). For such a function \( f \), by the definition of \( J, J \) it must be that \( f^J(\xi) = u \). For other functions \( f' \in c \) such that \( (f')^K(\xi') = (f')^I(\xi') \) for every \( \xi' \), as in Case 1, we conclude \( (f')^J(\xi) = (f')^I(\xi) \). Consequently, (C8) and (C7) both hold.

By Lemma 10, the fact \( J \models gr_f[F]^L_c \) follows from the assumption \( K \models gr_f[F]^L_c \).

**Theorem 5** For any e-plain sentence \( F \) of signature \( \sigma \), any list \( c \) of intensional constants, and any total interpretation \( I \) of \( \sigma \) satisfying \( \exists x y (x \neq y) \), \( I \models SM[F; c] \) iff \( I \models CBL[F; c] \).

**Proof.** We use Theorem 1 and Theorem 2 to refer to the grounding and reduct based definitions rather than the second-order logic based definitions. The claim follows from Lemma 11.

---

**C.5 Proof of Theorem 7 and Corollary 1**

**Lemma 12**

For any partial interpretation \( I \) and any atomic sentence \( p(t_1, \ldots, t_k) \) and \( f(t_1, \ldots, t_{k-1}) = t_k \),

(a) \( I \models_p p(t_1, \ldots, t_k) \) iff \( I \models_p \exists x_1 \cdots x_n (p(t_1, \ldots, t_k)^\prime = t_n_1 \land \cdots \land t_n_j = t_n_j) \)

where \( \{n_1, \ldots, n_j\} \subseteq \{1, \ldots, k\} \) and \( p(t_1, \ldots, t_k)^\prime \) is obtained from \( p(t_1, \ldots, t_k) \) by replacing each \( t_n_i \) in \( p(t_1, \ldots, t_k) \) with \( x_n_i \).

(b) \( I \models_p f(t_1, \ldots, t_{k-1}) = t_k \) iff \( I \models_p \exists x_1 \cdots x_n ((f(t_1, \ldots, t_{k-1}) = t_k)^\prime = t_n_1 \land \cdots \land t_n_j = t_n_j) \)

where \( \{n_1, \ldots, n_j\} \subseteq \{1, \ldots, k\} \) and \( (f(t_1, \ldots, t_{k-1}) = t_k)^\prime \) is obtained from \( f(t_1, \ldots, t_{k-1}) = t_k \) by replacing each \( t_n_i \) in \( f(t_1, \ldots, t_{k-1}) = t_k \) with \( x_n_i \).

**Proof.** Consider the following cases.

Case 1: \( t'_i = u \) for some \( i \in \{n_1, \ldots, n_j\} \). Clearly, \( I \models_p p(t_1, \ldots, t_k) \) and \( I \models_p f(t_1, \ldots, t_{k-1}) = t_k \). It is also the case that \( I \models_p t_i = \xi^\circ \) for any \( \xi \in |I| \) so we have

\[
I \models_p \exists x_1 \cdots x_n (p(t_1, \ldots, t_k)^\prime = t_n_1 \land \cdots \land t_n_j = t_n_j)
\]

and

\[
I \models_p \exists x_1 \cdots x_n ((f(t_1, \ldots, t_{k-1}) = t_k)^\prime = t_n_1 \land \cdots \land t_n_j = t_n_j).
\]

Case 2: \( t'_i = u \) for some \( i \in \{1, \ldots, k\} \setminus \{n_1, \ldots, n_j\} \). Clearly, \( I \models_p p(t_1, \ldots, t_k) \) and \( I \models_p f(t_1, \ldots, t_{k-1}) = t_k \). Also, since \( t_i \) remains in \( p(t_1, \ldots, t_k)^\prime \) and \( (f(t_1, \ldots, t_k) = t)^\prime \), we have

\[
I \models_p p(t_1, \ldots, t_k)^\prime \text{ and } I \models_p (f(t_1, \ldots, t_k) = t)^\prime,
\]

from which (C9) and (C10) follow.

Case 3: \( t'_i \neq u \) for all \( i \in \{1, \ldots, k\} \). Condition (a) clearly holds because it coincides with classical equivalence. For Condition (b), consider two subcases:

- **Subcase 1:** \( f(t_1, \ldots, t_{k-1}) = t \neq u \). Clearly, Condition (b) coincides with classical equivalence.
Case 1: \( f(t_1, \ldots, t_{k-1})^I = u \). Clearly, \( I \models | f(t_1, \ldots, t_{k-1}) = t_k \). Now in
\[
\exists x_{n_1} \ldots x_{n_j} ((f(t_1, \ldots, t_{k-1}) = t_k)^{\nu} \land t_{n_1} = x_{n_1} \land \cdots \land t_{n_j} = x_{n_j})
\]
there is only one set of values for \( x_{n_1} \ldots x_{n_j} \) that satisfies the last \( j \) conjunctive terms—when \( x_{n_1} \) is mapped to \( t_{n_1} \). However, for this set of values, \( ((f(t_1, \ldots, t_{k-1}))^{\nu})^I = f(t_1, \ldots, t_{k-1})^I = u \) (where \( (f(t_1, \ldots, t_{k-1}))^{\nu} \) is obtained from \( f(t_1, \ldots, t_{k-1}) \) by replacing each \( t_{n_i} \) with \( x_{n_i} \)) so (C10) holds.

Lemma 13
Given a sentence \( F \), a set of constants \( c \), and a partial interpretation \( I \), we have \( I \models | F \) iff \( I \models | UF_e(F) \).

Proof. The proof is by induction on the number of unfolding that needs to be done. More precisely, for any formula \( F \), we define \( NU_e(F) \) (“Needed Unfolding”) as follows.

- \( NU_e(p(t_1, \ldots, t_k)) = \)
  \[
  \begin{cases}
    0 & \text{if } p(t_1, \ldots, t_k) \text{ is } e\text{-plain;} \\
    \max(NU_e(t_1), \ldots, NU_e(t_k)) + 1 & \text{otherwise.}
  \end{cases}
  \]

- \( NU_e(f(t_1, \ldots, t_{k-1}) = t_k) = \)
  \[
  \begin{cases}
    0 & \text{if } f(t_1, \ldots, t_{k-1}) = t_k \text{ is } e\text{-plain;} \\
    \max(NU_e(t_1), \ldots, NU_e(t_k)) + 1 & \text{otherwise.}
  \end{cases}
  \]

- \( NU_e(G \circ H) = \max(NU_e(G), NU_e(H)) + 1 \), where \( \circ \in \{\land, \lor, \rightarrow\} \).

- \( NU_e(QxG) = NU_e(G) + 1 \), where \( Q \in \{\forall, \exists\} \).

Case 1: \( F \) is a \( c \)-plain atomic sentence. \( F \) is identical to \( UF_e(F) \) so the claim holds.

Case 2: \( F \) is \( p(t) \) where \( t \) contains at least one constant from \( c \). Let \( t_{n_1} \ldots t_{n_j} \) be the \( j \) terms in \( t \) containing at least one constant from \( c \). Now \( UF_e(F) \) is \( \exists x_{n_1} \ldots x_{n_j} (p(t_1, \ldots, t_k)^{\nu}_c \land UF_e(t_{n_1} = x_{n_1}) \land \cdots \land UF_e(t_{n_j} = x_{n_j})) \) where \( p(t_1, \ldots, t_k)^{\nu}_c \) is obtained from \( p(t_1, \ldots, t_k) \) by replacing each \( t_{n_i} \) in \( p(t_1, \ldots, t_k) \) with \( x_{n_i} \). Since \( NU_e(F) > NU_e(t_{n_i} = \xi^o) \) for each \( \xi \in |I| \) and each \( i \in \{1, \ldots, j\} \), by I.H. on \( t_{n_i} = \xi^o \), \( UF_e(t_{n_i} = x_{n_i}) \) can be replaced by \( t_{n_i} = x_{n_i} \) so that \( I \models | UF_e(F) \) iff \( I \models | \exists x_{n_1} \ldots x_{n_j} (p(t_1, \ldots, t_k)^{\nu}_c \land t_{n_1} = x_{n_1} \land \cdots \land t_{n_j} = x_{n_j}) \).

By Lemma 12 the latter is equivalent to \( I \models | F \).

Case 3: \( F \) is \( f(t) = t_1 \) where at least one of \( t \) and \( t_1 \) contain at least one constant from \( c \). Let \( t_{n_1} \ldots t_{n_j} \) be the \( j \) terms in \( t \) and \( t_1 \) containing at least one constant from \( c \). Now \( UF_e(F) \) is \( \exists x_{n_1} \ldots x_{n_j} ((f(t) = t_1)^{\nu}_c \land UF_e(t_{n_1} = x_{n_1}) \land \cdots \land UF_e(t_{n_j} = x_{n_j})) \), where \( (f(t) = t_1)^{\nu}_c \) is obtained from \( f(t) = t_1 \) by replacing each \( t_{n_i} \) in \( f(t) = t_1 \) with \( x_{n_i} \). Since \( NU_e(F) > NU_e(t_{n_i} = \xi^o) \) for each \( \xi \in |I| \) and each \( i \in \{1, \ldots, j\} \), by I.H. on \( t_{n_i} = \xi^o \), \( UF_e(t_{n_i} = x_{n_i}) \) can be replaced by \( t_{n_i} = x_{n_i} \) so that \( I \models | UF_e(F) \) iff \( I \models | \exists x_{n_1} \ldots x_{n_j} ((f(t) = t_1)^{\nu}_c \land t_{n_1} = x_{n_1} \land \cdots \land t_{n_j} = x_{n_j}) \).

By Lemma 12 the latter is equivalent to \( I \models | F \).

Case 4: \( F \) is \( G \circ H \) for \( \circ \in \{\land, \lor, \rightarrow\} \). By I.H. on \( G \) and \( H \).

Case 5: \( F \) is \( QxF(x) \) for \( Q \in \{\forall, \exists\} \). By I.H. on \( F(\xi^o) \) for each \( \xi \in |I| \).

Theorem 7 For any sentence \( F \), any list \( c \) of constants, and any partial interpretation \( I \), we have \( I \models | \text{CBL}[F; c] \) iff \( I \models | \text{CBL}[UF_e(F); c] \).
Theorem 6 For any head-plain sentence $F$ of signature $\sigma$ that is tight on $c$, and any total interpretation $I$ of $\sigma$ satisfying $\exists xy(x \neq y)$, we have $I \models SM[F; c]$ iff $I \models CBL[F; c]$.

Proof. We first note that since $F$ is head-plain and tight on $c$, we can transform this into Clark normal form that is still tight on $c$, so we can assume that $F$ is already turned into this form.

By Corollary 1, $I \models CBL[F; c]$ iff $I \models SM[UF_e(F); c]$, so it remains to check that $I \models SM[UF_e(F); c]$ iff $I \models SM[F; c]$.

It is easy to check that the completion of $UF_e(F)$ relative to $c$ is equivalent to the completion of $F$ relative to $c$. By Theorem 2 from (Bartholomew and Lee 2013), we conclude that $SM[UF_e(F); c]$ is equivalent to $SM[F; c]$.

C.7 Proof of Theorem 8, Corollary 2, and Corollary 3

Theorem 8 For any $f$-plain sentence $F$ and any partial interpretation $I$, if

$$I \models \forall xy(p(x, y) \leftrightarrow f(x) = y)$$

(C11)
then $I \models_{p} \text{CBL}[F; f, c]$ iff $I \models_{p} \text{CBL}[F^f_{\phi}; p, c]$.

**Proof.** For any partial interpretation $I$ of signature $\sigma \supseteq \{f, p, c\}$ satisfying (C11), it is clear that $I \models_{p} F$ iff $I \models_{p} F^f_{\phi}$ since $F^f_{\phi}$ is simply the result of replacing all $f(x) = y$ with $p(x, y)$. Thus it is sufficient to show that

$$I \models_{p} \exists \hat{\mathcal{C}} \left( \hat{\mathcal{F}} \right) \iff I \models_{p} \exists \mathcal{C} \left( \mathcal{F} \right).$$

**Left-to-right:** Assume $I \models_{p} \exists \hat{\mathcal{C}} \left( \hat{\mathcal{F}} \right)$, we wish to show that $I \models_{p} \hat{\mathcal{F}} \cap \mathcal{F}$. For any partial interpretation $I$ of signature $\sigma$, $J_{\{g, d\}}(\mathcal{F}) \cup I$ is an interpretation of the extended signature $\sigma' = \sigma \cup \{g, q\}$. We assume

$$J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (g, d) \prec (f, c) \wedge F^\dagger(g, d)$$

and wish to show that there is a predicate $q$ of the same arity as $f$ and any list of predicate and function constants $d$ that is similar to $c$. For any partial interpretation $\mathcal{I}$ of signature $\sigma$, $J_{\{g, d\}}(\mathcal{F}) \cup I$ is an interpretation of the extended signature $\sigma' = \sigma \cup \{g, q\}$. We assume

$$J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (g, d) \prec (f, c) \wedge F^\dagger (g, d)$$

We define the new predicate $q$ in terms of $g$ as follows:

$$q_{J_{\{g, d\}}(\mathcal{F}) \cup I} (\xi, \xi') = \begin{cases} \text{TRUE} & \text{if } g_{J_{\{g, d\}}(\mathcal{F}) \cup I} (\xi) = \xi' \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

We first show if $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (g, d) \prec (f, c)$ then $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (q, p, c)$. Since we assume $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (g, d) \prec (f, c)$, it follows that

$$J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} g = f,$$

and $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} d \prec c$. From (C11), (C12), and the definition of $q$, it follows that $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} q = p$. Consequently, $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (q, p, c)$. We now show that $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (F^\dagger(g, d) \delta_{1} (q, d))$ by proving $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} F^\dagger(g, d)$ iff $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (F^\dagger(g, d)).$

**Case 1:** $F$ is an $f$-plain atomic sentence of the form $p(t)$, or $t_1 = t_2$ such that $t_1$ does not contain $f$. The claim is obvious since $F^f_{\phi}$ is exactly $F$ and so $(F^f_{\phi})^{\dagger}(q, d)$ is exactly $F^\dagger(g, d)$.

**Case 2:** $F$ is an $f$-plain atomic sentence of the form $f(t) = t_1$. Then $F^\dagger(g, d)$ is $p(t')$, where $t'$ and $t_1'$ are obtained from $t$ and $t_1$ by replacing the members of $c$ with the corresponding members of $d$. $F^f_{\phi}$ is $p(t, t_1)$, and $(F^f_{\phi})^{\dagger}(q, d)$ is $q(t', t_1')$. From the definition of $q$, it follows that $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} g(t') = t_1' \iff q(t', t_1')$. Therefore, $J_{\{g, d\}}(\mathcal{F}) \cup I \models_{p} (F^\dagger(g, d))$.

**Case 3:** $F$ is $G \cap H$ where $\cap \in \{\land, \lor, \rightarrow\}$. By I.H. on $G$ and $H$.

**Case 4:** $F$ is $Q x G(x)$ where $Q \in \{\forall, \exists\}$. By I.H. on $G(\xi)$ for each $\xi \in I$. 


Right-to-left: Assume \( I \models_{p} \exists \hat{\sigma}((\hat{p}, \hat{c}) \prec (p, c) \land (F_{p}^{f})^{\dagger}(\hat{p}, \hat{c})) \). We wish to show that \( I \models_{p} \exists(\hat{f}, \hat{c})((\hat{f}, \hat{c}) \prec (f, c) \land F^{\dagger}(\hat{f}, \hat{c})) \). That is, take any predicate \( q \) of the same arity as \( p \) and any list of predicates and functions \( d \) that is similar to \( c \). As before, let \( J \) be a partial interpretation of \( \sigma \), and \( J_{(g,d)} \cup I \) is an interpretation of the extended signature \( \sigma' = \sigma \cup \{g, q, d\} \). We assume

\[
J_{(g,d)}^{(f,e)} \cup I \models_{p} (g, d) \prec (p, c) \land (F_{p}^{f})^{\dagger}(q, d)
\]

and wish to show that there is a function \( g \) of the same arity as \( f \) such that

\[
J_{(g,d)}^{(f,e)} \cup I \models_{p} (g, d) \prec (f, c) \land F^{\dagger}(g, d).
\]

We define \( g_{J_{(g,d)}^{(f,e)} \cup I}^{(f,e)} \) in terms of \( q \) as follows:

\[
g_{J_{(g,d)}^{(f,e)} \cup I}^{(f,e)}(\xi) = \begin{cases} 
F_{J_{(g,d)}^{(f,e)} \cup I}^{(f,e)}(\xi) & \text{if } q_{J_{(g,d)}^{(f,e)} \cup I}^{(f,e)}(\xi) = \text{TRUE} \\
\text{otherwise.} & 
\end{cases}
\]

We first show that if \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (q, d) \prec (p, c) \) then \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (g, d) \prec (f, c) \).

Case 1: \( J_{(g,d)}^{(f,e)} \cup I \models_{p} q = p \). Since we assume \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (q, d) \prec (p, c) \), it follows that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} d \prec c \). From (C11), \( J_{(g,d)}^{(f,e)} \cup I \models_{p} q = p \), and by the definition of \( g \), it follows that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} g = f \). Consequently, \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (g, d) \prec (f, c) \).

Case 2: \( J_{(g,d)}^{(f,e)} \cup I \models_{p} \neg(q = p) \). Since we assume \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (q, d) \prec (p, c) \), it follows that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} q \preceq p \) and so we have

\[
J_{(g,d)}^{(f,e)} \cup I \models_{p} q \prec p.
\]

From (C11), (C13), and the definition of \( g \), it follows that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} g \prec f \). Also from the assumption that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (q, d) \prec (p, c) \), it follows that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} d \prec c \). Consequently, \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (g, d) \prec (f, c) \).

We show that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} F^{\dagger}(g, d) \) by proving that \( J_{(g,d)}^{(f,e)} \cup I \models_{p} F^{\dagger}(q, d) \) iff \( J_{(g,d)}^{(f,e)} \cup I \models_{p} (F_{p}^{f})^{\dagger}(q, d) \). The proof is similar to the one above, and is omitted.

**Corollary 2.** Let \( F \) be an \( f \)-plain sentence. (a) For any partial interpretation \( I \) of the signature of \( F, I \models_{p} \text{CBL}[F; f, c] \iff I_{p}^{f} \models_{p} \text{CBL}[F_{p}^{f} \land UC_{p}; p, c] \). (b) For any partial interpretation \( J \) of the signature of \( F_{p}^{f} \), \( J \models_{p} \text{CBL}[F_{p}^{f} \land UC_{p}; p, c] \iff J = I_{p}^{f} \) for some partial interpretation \( I \) such that \( I \models_{p} \text{CBL}[F; f, c] \).

**Proof.** For two partial interpretations \( I \) of signature \( \sigma_{1} \) and \( J \) of signature \( \sigma_{2} \) with the same universe, by \( I \cup J \) we denote the partial interpretation of signature \( \sigma_{1} \cup \sigma_{2} \) that interprets all constants occurring only in \( \sigma_{1} \) in the same way as \( I \) does and similarly for \( \sigma_{2} \) and \( J \). For constants appearing in both \( \sigma_{1} \) and \( \sigma_{2} \), \( I \) must interpret these the same as \( J \) does, in which case \( I \cup J \) also interprets the constants in this way.

**Part (a). Left-to-right:** Assume \( I \models_{p} \text{CBL}[F; f, c] \). By the definition of \( I_{p}^{f} \), \( I \cup I_{p}^{f} \models_{p} (C11) \). Thus by Theorem 8, \( I \cup I_{p}^{f} \models_{p} \text{CBL}[F; f, c] \iff \text{CBL}[F_{p}^{f}; p, c] \). Since we assume \( I \models_{p} \text{CBL}[F; f, c] \), it is the case that \( I \cup I_{p}^{f} \models_{p} \text{CBL}[F; f, c] \) and thus it must be the case that \( I \cup I_{p}^{f} \models_{p} \text{CBL}[F_{p}^{f}; p, c] \).
Further, (C11) entails $UC_p$, so $I \cup I_p \models UC_p$. Since the signature of $I$ does not contain $p$, we conclude $I_p \models CBL[F_p \cup UC_p]$. Therefore, we assume $p \not\in UC_p$ and since $UC_p$ is comprised of constraints, $I_p \models CBL[F_p \cup UC_p; p, c]$.

Part (a), Right-to-left: Assume $I_p \models CBL[F_p \cup UC_p; p, c]$. By the definition of $I_p$, $I \cup I_p \models (C11)$. Thus by Theorem 8, $I \cup I_p \models CBL[F; f, c] \leftrightarrow CBL[F_p; p, c]$. From the assumption, we have $I_p \models CBL[F_p; p, c]$ and further $I \cup I_p \models CBL[F_p; p, c]$. Consequently, $I \cup I_p \models CBL[F; f, c]$ and since the signature of $I_p$ does not contain $f$, we conclude $I \models CBL[F; f, c]$.

Part (b), Left-to-right: Assume $J \models CBL[F_p \cup UC_p; p, c]$. Let $I = J^p$ where $J^p$ denotes the partial interpretation of the signature of $F$ obtained from $J$ by replacing the set $p'$ with the function $f$ such that $f'(\xi_1, \ldots, \xi_k) = \xi_{k+1}$ for all tuples $\langle \xi_1, \ldots, \xi_k, \xi_{k+1} \rangle$ in $p'$. This is a valid definition of a function since we assume $J \models CBL[F_p \cup UC_p; p, c]$, from which it follows that $J \models UC_p$. Clearly, $J = I_p$ so it only remains to be shown that $I \models CBL[F; f, c]$. By the definition of $J^p$, $I \cup J \models (C11)$. Thus by Theorem 8, $I \cup J \models CBL[F; f, c] \leftrightarrow CBL[F_p; p, c]$. From the assumption, we have $I \cup J \models CBL[F_p; p, c]$ and further $I \cup J \models CBL[F_p; p, c]$. Consequently, $I \cup J \models CBL[F; f, c]$ and since the signature of $J$ does not contain $f$, we conclude $I \models CBL[F; f, c]$.

Part (b), Right-to-left: Take any $I$ such that $J = I_p$ and $I \models CBL[F; f, c]$. By the definition of $J = I_p$, $I \cup J \models (C11)$. Thus by Theorem 8, $I \cup J \models CBL[F; f, c] \leftrightarrow CBL[F_p; p, c]$. Since we assume $I \models CBL[F; f, c]$, it is the case that $I \cup J \models CBL[F; f, c]$ and thus it must be the case that $I \cup J \models CBL[F_p; p, c]$. Further, (C11) entails $UC_p$, so $I \cup J \models UC_p$. Since the signature of $J$ does not contain $p$, we conclude $J \models CBL[F_p \cup UC_p; p, c]$ and since $UC_p$ is comprised of constraints, $J \models CBL[F_p \cup UC_p; p, c]$.

Corollary 3 Let $c$ be a set of intensional constants consisting of intensional function constants $f$ and intensional predicate constants, and let $F$ be an $c$-plain sentence. (a) For any total interpretation $I$ of the signature of $F$, $I \models CBL[F; c]$ iff $I_p \models SM[F_p \cup UC_p; c_E]$; (b) For any total interpretation $J$ of the signature of $F_p$, $J \models SM[F_p \cup UC_p; c_E]$ iff $J = I_p$ for some total interpretation $I$ such that $I \models CBL[F; c]$.

Proof. (a) First, by multiple applications of Corollary 2, it follows that for any total interpretation $I$ of the signature of $F$, $I \models CBL[F; c]$ iff $I_p \models CBL[F_p \cup UC_p; c_E]$. Then the statement follows from Theorem 5 since $F_p \cup UC_p$ is $c$-plain.

The proof of (b) is similar.

C.8 Proof of Theorem 9

Given a program $\Pi$, by $\Pi^{POL}$ we denote the FOL representation of $\Pi$.

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3 The last step is justified by the theorem on constraints, similar to Theorem 3 from (Ferraris et al. 2011), which we omit here.
Lemma 14
Consider a signature $\sigma$ and a set of constants $c$. Given an ASP\{f\} program $\Pi$ of signature $\sigma$ not containing strong negation,

(a) For any partial interpretation $I$ of signature $\sigma$ that maps every constant in $\sigma \setminus c$ to itself, there is a consistent set $S$ of seed literals such that $I \models_{\Pi}^{\text{FOL}} S \models_{\Pi}^{\text{FOL}} \Pi$.
(b) For any consistent set of seed literals $S$, there is a partial interpretation $I$ such that $I \models_{\Pi}^{\text{FOL}} S \models_{\Pi}^{\text{FOL}} \Pi$.

Proof. Part (a): Given a partial interpretation $I$, let $S$ be the set $\{f(v) = w : f(v)^I = w\} \cup \{p(v) : p(v)^I = \text{TRUE}\}$. We note that this is a consistent set of seed literals since a partial interpretation maps $f(v)$ to at most one object constant.

We also note that by the definition of $S$, for any atomic sentence $A$, we have $I \models_{\Pi}^{\text{FOL}} A$ iff $S \models_{\Pi}^{\text{FOL}} A$. Now, consider any rule $r$ from $\Pi$. $I \models_{\Pi}^{\text{FOL}} r$ iff $I \models_{\Pi}^{\text{FOL}} \text{head}(r)$ or $I \not\models_{\Pi}^{\text{FOL}} \text{body}(r)$. By the previous observation, this is equivalent to $S \models_{\Pi}^{\text{FOL}} \text{head}(r)$ or $S \not\models_{\Pi}^{\text{FOL}} \text{body}(r)$ since $\text{body}(r)$ is a conjunction of atomic formulas. This is precisely the definition of $S \models_{\Pi}^{\text{FOL}} r$.

Part (b): Given a consistent set of seed literals $S$, let $I$ be the partial interpretation defined as follows:

- for every object constant $v \in \sigma \setminus c$, we have $v^I = v$.
- for every predicate constant $p \in c$ and every list of object constants $v$, we have $p(v)^I = \text{TRUE}$ iff $p(v) \in S$.
- for every function constant $f \in c$ and every list of object constants $v$, we have $f(v)^I = u$ if $S$ does not mention $f(v)$, and $f(v)^I = w$ if $f(v) = w$ is in $S$.

We note that the last bullet is well-defined since $S$ is a consistent set of seed literals so that there cannot be two distinct object constants $a$ and $b$ such that $f(v) = a \in S$ and $f(v) = b \in S$.

We also note that by the definition of $I$, for any atomic sentence $A$, we have $I \models_{\Pi}^{\text{FOL}} A$ iff $S \models_{\Pi}^{\text{FOL}} A$. Now, consider any rule $r$ from $\Pi$. $S \models_{\Pi}^{\text{FOL}} r$ iff $S \models_{\Pi}^{\text{FOL}} \text{head}(r)$ or $S \not\models_{\Pi}^{\text{FOL}} \text{body}(r)$. By the previous observation, this is equivalent to $I \models_{\Pi}^{\text{FOL}} \text{head}(r)$ or $I \not\models_{\Pi}^{\text{FOL}} \text{body}(r)$ since $\text{body}(r)$ is a conjunction of atomic formulas. This is precisely the definition of $I \models_{\Pi}^{\text{FOL}} r$.

The proof of Lemma 14 tells us that a consistent set of seed literals can be identified with a partial interpretation.

Lemma 15
For consistent sets of seed literals $J$ and $I$ of the same signature, $J$ is a proper subset of $I$ iff $J \prec^c I$ (as defined in Section 2.3.2) when we view them as partial interpretations.

Proof. We first note that since consistent sets of literals map every object constant in $\sigma \setminus c$ to itself, the partial interpretation view does the same which corresponds to the first condition for $J \prec^c I$. The second condition of $J \prec^c I$ is $p^J \subseteq p^I$ for all predicate constants in $c$, which corresponds exactly to the predicate part of $J$ being a subset of the predicate part of $I$. Finally, the third condition of $J \prec^c I$ is $f^J(\xi) = u$ or $f^J(\xi) = f^I(\xi)$ corresponds to the function part of $J$ being a subset of the function part of $I$ since we identify a partial interpretation mapping an element to $u$ to the absence of that element in the set.

Theorem 9 For any ASP\{f\} program $\Pi$ with intensional constants $c$ and any consistent set $I$ of seed literals, if $\Pi$ has no strong negation, then $I$ is a Balduccini answer set of $\Pi$ iff $I \models_{\Pi}^{\text{CBL}[\Pi; c]}$.  


Proof. By definition and by using the equivalent reformulation presented and justified in Lemma 15 and Lemma 14, \( I \) is a Balduccini answer set of a program \( \Pi \) iff \( I \models v \Pi \) and for every partial interpretation \( J \) such that \( J \prec_c I \), we have \( J \not\models v \Pi^I \). This is equivalent to the reduct reformulation of the Cabalar semantics. Further, this is equivalent to \( I \models v \text{CBL}[\Pi^F; c] \) by Theorem 2.

C.9 Proof of Theorem 10

**Theorem 10** For any ASP(\( f \)) program \( \Pi \) with intensional constants \( c \) and any consistent set \( I \) of seed literals, \( I \) is a Balduccini answer set of \( \Pi \) iff \( I \) is a Balduccini answer set of \( \Pi^# \).

**Proof.** First, we show that \( I \models \sim(f = g) \iff I \models (f = f) \land (g = g) \land \neg(f = g) \).

Left-to-right: Assume \( I \models \sim(f = g) \). By definition, \( I \) contains both \( f = c_1 \) and \( g = c_2 \) for some object constants \( c_1 \) and \( c_2 \) such that \( c_1 \neq c_2 \). Clearly, each of \( I \models f = f \), \( I \models g = g \) and \( I \not\models f = g \) holds.

Right-to-left: \( I \models (f = f) \land (g = g) \land \neg(f = g) \). Since \( I \models f = f \) and \( I \models g = g \), it follows that \( I \) contains \( f = c_1 \) and \( I \) contains \( f = c_2 \) for some \( c_1 \) and \( c_2 \). Further, since \( I \models \neg(f = g) \), it must be that \( c_1 \neq c_2 \), from which the claim follows.

From this it is not difficult to check that \( \Pi^I \) is equivalent to \( (\Pi^#)^I \) under partial satisfaction, from which the claim follows.

C.10 Proof of Theorem 11

**Theorem 11** For any sentence \( F \) in Clark normal form that is tight on \( c \) and any total interpretation \( I \), if \( I \models \exists xy(x \neq y) \), then \( I \models v \text{CBL}[F; c] \iff I \models v \text{SM}[F; c] \iff I \) is a model of the completion of \( F \) relative to \( c \).

**Proof.** By Theorem 2 from (Bartholomew and Lee 2013), \( I \) is a model of the completion of \( F \) relative to \( c \) iff \( I \models v \text{SM}[F; c] \). Since a formula in Clark normal form that is tight on \( c \) is also head-c-plain and is tight on \( c \), \( I \models v \text{SM}[F; c] \iff I \models v \text{CBL}[F; c] \) by Theorem 6.

References
