Online appendix for the paper

On the Stable Model Semantics for Intensional Functions

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Appendix A Completion and the Cabalar Semantics

The following definitions are from (Bartholomew and Lee 2013).

We say that a formula F is in *Clark normal form* (relative to a list $\mathbf{c}$ of intensional constants) if it is a conjunction of sentences of the form

$$\forall \mathbf{x}(G \rightarrow p(\mathbf{x})) \tag{A1}$$

and

$$\forall \mathbf{x}y(G \rightarrow f(\mathbf{x})\!=\!y) \tag{A2}$$

one for each intensional predicate p and each intensional function f, where $\mathbf{x}$ is a list of distinct object variables, y is an object variable, and G is an arbitrary formula that has no free variables other than those in $\mathbf{x}$ and y.

The *completion* of a formula F in Clark normal form (relative to $\mathbf{c}$) is obtained from F by replacing each conjunctive term (A1) with

$$\forall \mathbf{x}(p(\mathbf{x}) \leftrightarrow G)$$

and each conjunctive term (A2) with

$$\forall \mathbf{x}y(f(\mathbf{x})\!=\!y \leftrightarrow G).$$

An occurrence of a symbol or a subformula in a formula F is called *strictly positive* in F if that occurrence is not in the antecedent of any implication in F. The *dependency graph* of F (relative to $\mathbf{c}$) is the directed graph that

- has all members of $\mathbf{c}$ as its vertices, and
- has an edge from c to d if, for some strictly positive occurrence of $G \rightarrow H$ in F,
  - — c has a strictly positive occurrence in H, and
  - — d has a strictly positive occurrence in G.

We say that F is *tight* (on $\mathbf{c}$) if the dependency graph of F (relative to $\mathbf{c}$) is acyclic.

The following theorem relates the Cabalar semantics to completion, which follows immediately from Theorem 12 from (Bartholomew and Lee 2013) and Theorem 6.

Theorem 11
For any sentence F in Clark normal form that is tight on $\mathbf{c}$ and any total interpretation I, if $I \models \exists xy(x \neq y)$, then $I \models_{\rm p} \text{CBL}[F; \mathbf{c}]$ iff $I \models \text{SM}[F; \mathbf{c}]$ iff I is a model of the completion of F relative to $\mathbf{c}$.

Appendix B Review of the Balduccini Semantics

The following is a review of the Balduccini semantics. Let us restrict a signature σ to be comprised of a set of *intensional* function and predicate constants denoted $\mathbf{c}$ as well as a set of *non-intensional* object constants $\sigma \setminus \mathbf{c}$.

Balduccini considered *terms* to have the form $f(c_1, \dots, c_k)$ where f is an intensional function constant (in $\mathbf{c}$), and each c_i is a non-intensional object constant (in $\sigma \setminus \mathbf{c}$). He considered an *atom* to be an expression $p(c_1, \dots, c_k)$ where p is an intensional predicate constant, and each c_i is a non-intensional object constant; a *t-atom* is an expression of the form $f = g$ where f is a term and g is either a term or a non-intensional object constant; a *seed t-atom* is a t-atom of the form $f = c$ where c is a non-intensional object constant. A *t-literal* is a t-atom $f = g$ or $\sim(f = g)$, where $\sim$ denotes strong negation. A *seed literal* is an atom a, or $\sim a$, or a seed t-atom. A *literal* is an atom a, or $\sim a$, or a t-literal. An ASP{f} program consists of rules of the form

$$h \leftarrow l_1, \dots, l_m, not\ l_{m+1}, \dots, not\ l_n\ , \tag{B1}$$

where h is a seed literal or $\bot$, and each l_i is a literal. An ASP{f} program is a finite set of rules. We identify rule (B1) with an implication

$$l_1 \wedge \dots \wedge l_m \wedge \neg l_{m+1} \wedge \dots \wedge \neg l_n \rightarrow h\ ,$$

and an ASP{f} program as the conjunction of all rules in it. Note that ASP{f} programs do not contain variables, and can be viewed as a special case of head-$\mathbf{c}$-plain formulas.

A set I of seed literals is said to be *consistent* if it contains no pair of an atom a and its strong negation $\sim a$; and contains no pair of seed t-atoms $t = c_1$ and $t = c_2$ such that $c_1 \neq c_2$. It is clear that any subset of a consistent set of seed literals is consistent as well.

The notion of satisfaction between a consistent set I of seed literals and literals, denoted by $\models_{\rm b}$, is defined as follows.

- For a seed literal l, $I \models_{\rm b} l$ if $l \in I$;
- For a non-seed literal $f = g$, $I \models_{\rm b} f = g$ if I contains both $f = c$ and $g = c$ for some object constant c;
- For a non-seed literal $\sim(f = g)$, $I \models_{\rm b} \sim(f = g)$ if I contains both $f = c_1$ and $g = c_2$ for some object constants c_1 and c_2 such that $c_1 \neq c_2$.

This notion of satisfaction is extended to formulas allowing $\wedge$, $\neg$ and $\leftarrow$ as in classical logic.

The reduct of an ASP{f} program Π relative to a consistent set I of seed literals is denoted $\Pi^{\underline{I}}$ and is defined as

$$\Pi^{\underline{I}} = \{h \leftarrow l_1\ \dots, l_m \mid \text{(B1)} \in \Pi \text{ and } I \models \neg l_{m+1} \wedge \dots \wedge \neg l_n\}\ .$$

I is called a *Balduccini answer set* of Π if

- $I \models_{\rm b} \Pi^{\underline{I}}$, and,
- for every proper subset J of I, we have $J \not\models_{\rm b} \Pi^{\underline{I}}$.

Appendix C Proofs

C.1 Proof of Theorem 1

We will often use the following notation. Let σ be a first-order signature, let $\mathbf{c}$ be a set of constants that is a subset of σ, and let $\mathbf{d}$ be a set of constants not belonging to σ and is similar to $\mathbf{c}$.[1] $J^{\mathbf{c}}_{\mathbf{d}}$ denotes the interpretation of signature $(\sigma \setminus \mathbf{c}) \cup \mathbf{d}$ obtained from J by replacing every constant from $\mathbf{c}$ with the corresponding constant from $\mathbf{d}$. For two interpretations I and J of σ that agree on all constants in $\sigma \setminus \mathbf{c}$, we define $J^{\mathbf{c}}_{\mathbf{d}} \cup I$ to be the interpretation from the extended signature $\sigma \cup \mathbf{d}$ such that

- $J^{\mathbf{c}}_{\mathbf{d}} \cup I$ agrees with I on all constants in $\mathbf{c}$;
- $J^{\mathbf{c}}_{\mathbf{d}} \cup I$ agrees with $J^{\mathbf{c}}_{\mathbf{d}}$ on all constants in $\mathbf{d}$;
- $J^{\mathbf{c}}_{\mathbf{d}} \cup I$ agrees with both I and J on all constants in $\sigma \setminus \mathbf{c}$.

Lemma 1
For any sentence F of signature σ and any interpretations I and J of σ,

(a) if $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models F^*(\mathbf{d})$, then $I \models F$;
(b) if $\langle J, I\rangle \models_{\text{fht}} F$, then $\langle I, I\rangle \models_{\text{fht}} F$.

Proof. By induction on F. ∎

Lemma 2
Let F be a sentence of signature σ, and let I and J be interpretations of σ such that $J <^{\mathbf{c}} I$. We have $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models F^*(\mathbf{d})$ iff $J \models gr_I[F]^{\underline{I}}$.

Proof. By induction on F.

Case 1: F is an atomic sentence. Then $F^*(\mathbf{d})$ is $F(\mathbf{d}) \wedge F$, where $F(\mathbf{d})$ is obtained from F by replacing the members of $\mathbf{c}$ with the corresponding members of $\mathbf{d}$. Consider the following subcases:

- *Subcase 1:* $I \not\models F$. Then $J^{\mathbf{c}}_{\mathbf{d}} \cup I \not\models F^*(\mathbf{d})$. Further, $gr_I[F]^{\underline{I}} = \bot$, so $J \not\models gr_I[F]^{\underline{I}}$.
- *Subcase 2:* $I \models F$. Then $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models F^*(\mathbf{d})$ iff $J^{\mathbf{c}}_{\mathbf{d}} \models F(\mathbf{d})$ iff $J \models F$. Further, $gr_I[F]^{\underline{I}} = F$, so $J \models gr_I[F]^{\underline{I}}$ iff $J \models F$.

Case 2: F is $G \wedge H$ or $G \vee H$. The claim follows immediately from I.H. on G and H.

Case 3: F is $G \rightarrow H$. Then $F^*(\mathbf{d}) = (G^*(\mathbf{d}) \rightarrow H^*(\mathbf{d})) \wedge (G \rightarrow H)$. Consider the following subcases:

- *Subcase 1:* $I \not\models G \rightarrow H$. Then $J^{\mathbf{c}}_{\mathbf{d}} \cup I \not\models F^*(\mathbf{d})$. Further, $gr_I[F]^{\underline{I}} = \bot$, which J does not satisfy.
- *Subcase 2:* $I \models G \rightarrow H$. Then $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models F^*(\mathbf{d})$ iff $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models G^*(\mathbf{d}) \rightarrow H^*(\mathbf{d})$. On the other hand, $gr_I[F]^{\underline{I}} = gr_I[G]^{\underline{I}} \rightarrow gr_I[H]^{\underline{I}}$ so this case holds by I.H. on G and H.

Case 4: F is $\exists x G(x)$. By I.H., $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models G(\xi^\diamond)^*(\mathbf{d})$ iff $J \models gr_I[G(\xi^\diamond)]^{\underline{I}}$ for each $\xi \in |I|$. The claim follows immediately.

Case 5: F is $\forall x G(x)$. Similar to Case 4. ∎

[1] That is to say, $\mathbf{d}$ and $\mathbf{c}$ have the same length and the corresponding members are either predicate constants of the same arity or function constants of the same arity.

Lemma 3
For any interpretations I and J of signature σ, we have $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models \mathbf{d} < \mathbf{c}$ iff $J <^{\mathbf{c}} I$.

Proof. Recall that by definition, $\mathbf{d} < \mathbf{c}$ is

$$(\mathbf{d}^{pred} \leq \mathbf{c}^{pred}) \wedge \neg(\mathbf{d} = \mathbf{c}),$$

and by definition, $J <^{\mathbf{c}} I$ is

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants p in $\mathbf{c}$; and
- J and I do not agree on $\mathbf{c}$.

First, by the definition of $J^{\mathbf{c}}_{\mathbf{d}} \cup I$, J and I have the same universe and agree on all constants in $\sigma \setminus \mathbf{c}$.

Second, by definition, $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models \mathbf{d}^{pred} \leq \mathbf{c}^{pred}$ iff, for every predicate constant p in $\mathbf{c}$,

$$J^{\mathbf{c}}_{\mathbf{d}} \cup I \models \forall \mathbf{x}(p(\mathbf{x})^{\mathbf{c}}_{\mathbf{d}} \rightarrow p(\mathbf{x})),$$ [2]

which is equivalent to saying that $(p^{\mathbf{c}}_{\mathbf{d}})^{J^{\mathbf{c}}_{\mathbf{d}} \cup I} \subseteq p^{J^{\mathbf{c}}_{\mathbf{d}} \cup I}$. Since I does not interpret any constant from $\mathbf{d}$, and $J^{\mathbf{c}}_{\mathbf{d}}$ does not interpret any constant from $\mathbf{c}$, this is equivalent to $(p^{\mathbf{c}}_{\mathbf{d}})^{J^{\mathbf{c}}_{\mathbf{d}}} \subseteq p^I$ and further to $p^J \subseteq p^I$.

Third, since I does not interpret any constant from $\mathbf{d}$ and $J^{\mathbf{c}}_{\mathbf{d}}$ does not interpret any constant from $\mathbf{c}$, $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models \neg(\mathbf{d} = \mathbf{c})$ is equivalent to saying that J and I do not agree on $\mathbf{c}$. ∎

Theorem 1 Let F be a first-order sentence of signature σ and $\mathbf{c}$ be a list of intensional constants. For any interpretation I of σ, $I \models \mathrm{SM}[F; \mathbf{c}]$ iff

- *I satisfies F, and*
- *every interpretation J such that $J <^{\mathbf{c}} I$ does not satisfy $(gr_I[F])^{\underline{I}}$.*

Proof. $I \models \mathrm{SM}[F; \mathbf{c}]$ is by definition

$$I \models F \wedge \neg \exists \widehat{\mathbf{c}}(\widehat{\mathbf{c}} < \mathbf{c} \wedge F^*(\widehat{\mathbf{c}})). \tag{C1}$$

The first item, "I satisfies F", is equivalent to the first conjunctive term of (C1).

By Lemma 2 and Lemma 3, the second item, "no interpretation J of σ such that $J <^{\mathbf{c}} I$ satisfies $gr_I[F]^{\underline{I}}$", is equivalent to the second conjunctive term in (C1). ∎

C.2 Proofs of Theorem 2 and Theorem 3

Recall the definition: $J \preceq^{\mathbf{c}} I$ if

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants in $\mathbf{c}$; and
- $f^J(\boldsymbol{\xi}) = u$ or $f^J(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ for all function constants in $\mathbf{c}$ and all lists $\boldsymbol{\xi}$ of elements in the universe.

As before, let $\mathbf{d}$ be a list of constants that is similar to $\mathbf{c}$ and is disjoint from σ. The notion of $J^{\mathbf{c}}_{\mathbf{d}} \cup I$ is straightforwardly extended to the case when J and I are partial interpretations.

[2] $p(\mathbf{x})^{\mathbf{c}}_{\mathbf{d}}$ denotes the atom that is obtained from $p(\mathbf{x})$ by replacing p with the corresponding member of $\mathbf{d}$ if $p \in \mathbf{c}$, and no change otherwise.

Lemma 4
For any partial interpretations I and J of signature σ, we have $J \preceq^{\mathbf{c}} I$ iff $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \mathbf{d} \preceq \mathbf{c}$.

Proof. By the definition of $J^{\mathbf{c}}_{\mathbf{d}} \cup I$, J and I have the same universe and agree on all constants in $\sigma \setminus \mathbf{c}$, which is the first condition of $J \preceq^{\mathbf{c}} I$.

Recall the definition: $\mathbf{d} \preceq \mathbf{c}$ is

$$(\mathbf{d}^{pred} \leq \mathbf{c}^{pred}) \wedge (\mathbf{d}^{func} \leq \mathbf{c}^{func}).$$

$J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \mathbf{d}^{pred} \leq \mathbf{c}^{pred}$ iff, for every predicate constant p in $\mathbf{c}$,

$$J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \forall \mathbf{x}(p(\mathbf{x})^{\mathbf{c}}_{\mathbf{d}} \rightarrow p(\mathbf{x})),$$

which is equivalent to saying that $(p^{\mathbf{c}}_{\mathbf{d}})^{J^{\mathbf{c}}_{\mathbf{d}} \cup I} \subseteq p^{J^{\mathbf{c}}_{\mathbf{d}} \cup I}$. Since I does not interpret any constant from $\mathbf{d}$ and $J^{\mathbf{c}}_{\mathbf{d}}$ does not interpret any constant from $\mathbf{c}$, this is equivalent to $(p^{\mathbf{c}}_{\mathbf{d}})^{J^{\mathbf{c}}_{\mathbf{d}}} \subseteq p^I$ and further to $p^J \subseteq p^I$, which is the second condition of $J \preceq^{\mathbf{c}} I$.

$J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} (\mathbf{d}^{func} \leq \mathbf{c}^{func})$ iff, for every function constant f in $\mathbf{c}$,

$$J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \forall \mathbf{x}((f(\mathbf{x})^{\mathbf{c}}_{\mathbf{d}} \neq f(\mathbf{x})^{\mathbf{c}}_{\mathbf{d}}) \vee (f(\mathbf{x})^{\mathbf{c}}_{\mathbf{d}} = f(\mathbf{x}))),$$

which is equivalent to saying that $f^J(\boldsymbol{\xi}) = u$ or $f^J(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$, the third condition of $J \preceq^{\mathbf{c}} I$. ∎

Lemma 5
For any partial interpretations I and J of signature σ, we have $J \prec^{\mathbf{c}} I$ iff $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \mathbf{d} \prec \mathbf{c}$.

Proof. Immediate from Lemma 4 since

- $J \prec^{\mathbf{c}} I$ iff $J \preceq^{\mathbf{c}} I$ and not $I \preceq^{\mathbf{c}} J$, and
- $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \mathbf{d} \prec \mathbf{c}$ iff $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \mathbf{d} \preceq \mathbf{c}$ and $J^{\mathbf{c}}_{\mathbf{d}} \cup I \not\models_{\mathrm{p}} \mathbf{c} \preceq \mathbf{d}$.

∎

Lemma 6
For any sentence F of signature σ and any partial interpretations I and J of σ such that $J \preceq^{\mathbf{c}} I$,

(a) if $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} F^{\dagger}(\mathbf{d})$, then $I \models_{\mathrm{p}} F$;
(b) if $\langle J, I \rangle \models_{\mathrm{pht}} F$, then $\langle I, I \rangle \models_{\mathrm{pht}} F$.

Proof. Each of (a) and (b) can be proved by induction on F.

We will show only the case when F is an atomic sentence. The other cases are straightforward:
Part (a): Let F be an atomic sentence. Assume $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} F^{\dagger}(\mathbf{d})$, i.e., $J \models_{\mathrm{p}} F$.

- *Subcase 1:* F is of the form $p(\mathbf{t})$. Since $J \preceq^{\mathbf{c}} I$, it follows that $I \models_{\mathrm{p}} F$.
- *Subcase 2:* F is of the form $t_1 = t_2$. Since $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} F(\mathbf{d})$, $t_1^J = t_2^J \neq u$. From $J \preceq^{\mathbf{c}} I$, it follows that $t_1^I = t_2^I \neq u$, i.e., $I \models_{\mathrm{p}} F$.

Part (b): Let F be an atomic sentence. Assume $\langle J, I \rangle \models_{\mathrm{pht}} F$, i.e., $\langle J, I \rangle, h \models_{\mathrm{pht}} F$

- *Subcase 1:* F is of the form $p(\mathbf{t})$. Since $J \preceq^{\mathbf{c}} I$, it follows that $\langle J, I \rangle, t \models_{\mathrm{pht}} F$.
- *Subcase 2:* F is of the form $t_1 = t_2$. Since $\langle J, I \rangle, h \models_{\mathrm{pht}} F$, $t_1^J = t_2^J \neq u$. From $J \preceq^{\mathbf{c}} I$, it follows that $t_1^I = t_2^I \neq u$, i.e., $\langle J, I \rangle, t \models_{\mathrm{pht}} F$.

∎

Lemma 7
Let F be a sentence of signature σ, and let I and J be partial interpretations of σ such that $J \preceq^{\mathbf{c}} I$. We have $J \models_{\rm p} gr_I[F]^{\underline{I}}$ iff $\langle J, I\rangle \models_{\rm pht} F$.

Proof. By induction on F.

Case 1: F is an atomic sentence. Clearly, $gr_I[F]$ is F.

- *Subcase 1:* $I \not\models_{\rm p} F$. Then $gr_I[F]^{\underline{I}}$ is $\bot$, and $J \not\models_{\rm p} \bot$. Further, since $\langle I, I\rangle \not\models_{\rm pht} F$, by Lemma 6 (b), it follows that $\langle J, I\rangle \not\models_{\rm pht} F$.
- *Subcase 2:* $I \models_{\rm p} F$. Then $gr_I[F]^{\underline{I}}$ is F. It is clear that $J \models_{\rm p} F$ iff $\langle J, I\rangle \models_{\rm pht} F$.

Case 2: F is $G \wedge H$ or $G \vee H$. The claim follows immediately from I.H. on G and H.

Case 3: F is $G \rightarrow H$. Consider the following subcases:

- *Subcase 1:* $I \not\models_{\rm p} G \rightarrow H$. $gr_I[G \rightarrow H]^{\underline{I}}$ is $\bot$, and $J \not\models_{\rm p} \bot$. Further, $\langle I, I\rangle \not\models_{\rm p} G \rightarrow H$. By Lemma 6 (b), $\langle J, I\rangle \not\models_{\rm p} G \rightarrow H$.
- *Subcase 2:* $I \models_{\rm p} G \rightarrow H$. $gr_I[G \rightarrow H]^{\underline{I}}$ is equivalent to $gr_I[G]^{\underline{I}} \rightarrow gr_I[H]^{\underline{I}}$. Further, $\langle J, I\rangle \models_{\rm pht} G \rightarrow H$ is equivalent to saying that $\langle J, I\rangle \not\models_{\rm pht} G$ or $\langle J, I\rangle \models_{\rm pht} H$. Then the claim follows from I.H. on G and H.

Case 4: F is $\forall x G(x)$, or $\exists x G(x)$. By induction on $G(\xi^{\diamond})$ for each ξ in the universe. ∎

Theorem 2 *Let F be a first-order sentence of signature σ and let $\mathbf{c}$ be a list of intensional constants. For any partial interpretation I of σ, $\langle I, I\rangle$ is a partial equilibrium model of F iff*

- *$I \models_{\rm p} F$, and*
- *for every partial interpretation J of σ such that $J \prec^{\mathbf{c}} I$, we have $J \not\models_{\rm p} gr_I[F]^{\underline{I}}$.*

Proof. Clearly, $I \models_{\rm p} F$ iff $\langle I, I\rangle \models_{\rm pht} F$. By Lemma 7, for every partial interpretation J of σ such that $J \prec^{\mathbf{c}} I$, $J \not\models_{\rm p} gr_I[F]^{\underline{I}}$ iff $\langle J, I\rangle \not\models_{\rm pht} F$. ∎

Lemma 8
Let F be a sentence of signature σ, and let I and J be partial interpretations of σ. We have $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\rm p} F^{\dagger}(\mathbf{d})$ iff $\langle J, I\rangle \models_{\rm pht} F$.

Proof. By induction on F.

Case 1: F is an atomic sentence. $F^{\dagger}(\mathbf{d})$ is $F(\mathbf{d})$. $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\rm p} F(\mathbf{d})$ iff $J \models_{\rm p} F$ iff $\langle J, I\rangle, h \models_{\rm pht} F$ iff $\langle J, I\rangle \models_{\rm pht} F$.

Case 2: F is $G \wedge H$ or $G \vee H$. Follows by I.H. on G and H.

Case 3: F is $G \rightarrow H$. Consider the following subcases:

- *Subcase 1:* $I \not\models_{\rm p} G \rightarrow H$. Clearly, $J^{\mathbf{c}}_{\mathbf{d}} \cup I \not\models_{\rm p} G \rightarrow H$ and $\langle J, I\rangle \not\models_{\rm pht} G \rightarrow H$.
- *Subcase 2:* $I \models_{\rm p} G \rightarrow H$. Then $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\rm p} (G \rightarrow H)^{\dagger}(\mathbf{d})$ iff $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\rm p} G^{\dagger}(\mathbf{d}) \rightarrow H^{\dagger}(\mathbf{d})$. Further, $\langle J, I\rangle \models_{\rm pht} G \rightarrow H$ is equivalent to saying that $\langle J, I\rangle \not\models_{\rm pht} G$ or $\langle J, I\rangle \models_{\rm pht} H$. Then the claim follows from I.H. on G and H.

Case 4: F is $\forall x G(x)$, or $\exists x G(x)$. By induction on $G(\xi^\diamond)$ for each ξ in the universe. ∎

Theorem 3 For any sentence F, a PHT-interpretation $\langle I, I\rangle$ is a partial equilibrium model of F relative to $\mathbf{c}$ *iff* $I \models_{\rm p} \text{CBL}[F; \mathbf{c}]$.

Proof. By definition, $\text{CBL}[F; \mathbf{c}]$ is

$$F \wedge \neg\exists\widehat{\mathbf{c}}(\widehat{\mathbf{c}} \prec \mathbf{c} \wedge F^\dagger(\widehat{\mathbf{c}})).$$

Clearly, $I \models_{\rm p} F$ iff $\langle I, I\rangle \models_{\rm pht} F$. From Lemma 5 and Lemma 8, it follows that $I \models_{\rm p} \neg\exists\widehat{\mathbf{c}}(\widehat{\mathbf{c}} \prec \mathbf{c} \wedge F^\dagger(\widehat{\mathbf{c}}))$ iff there is no interpretation J of σ such that $J \prec^{\mathbf{c}} I$ and $\langle J, I\rangle \models_{\rm pht} F$. ∎

C.3 Proof of Theorem 4

Lemma 9
Let F be a sentence of signature σ and let I and J be interpretations of σ such that $J <^{\mathbf{c}} I$. We have $J \models gr_I[F]^{\underline{I}}$ iff $\langle J, I\rangle \models_{\rm fht} F$.

Proof. By induction on F.

Case 1: F is an atomic sentence. $gr_I[F]$ is F.

- *Subcase 1:* $I \not\models F$. Then $gr_I[F]^{\underline{I}}$ is $\bot$, which J does not satisfy. Further, since $\langle J, I\rangle, t \not\models_{\rm fht} F$, $\langle J, I\rangle \not\models_{\rm fht} F$.
- *Subcase 2:* $I \models F$. Then $gr_I[F]^{\underline{I}}$ is F, and $\langle J, I\rangle, t \models_{\rm fht} F$. It is clear that $J \models F$ iff $\langle J, I\rangle, h \models_{\rm fht} F$.

Case 2: F is $G \wedge H$ or $G \vee H$. The claim follows immediately from I.H. on G and H.

Case 3: F is $G \rightarrow H$. Consider the following subcases:

- *Subcase 1:* $I \not\models G \rightarrow H$. Then $gr_I[G \rightarrow H]^{\underline{I}}$ is $\bot$, which J does not satisfy. Further, $\langle I, I\rangle \not\models_{\rm fht} G \rightarrow H$. By Lemma 1 (b), $\langle J, I\rangle \not\models_{\rm fht} G \rightarrow H$.
- *Subcase 2:* $I \models G \rightarrow H$. Then $gr_I[G \rightarrow H]^{\underline{I}}$ is equivalent to $gr_I[G]^{\underline{I}} \rightarrow gr_I[H]^{\underline{I}}$. Further, $\langle J, I\rangle \models_{\rm fht} G \rightarrow H$ is equivalent to saying that $\langle J, I\rangle \not\models_{\rm fht} G$ or $\langle J, I\rangle \models_{\rm fht} H$. Then the claim follows from I.H. on G and H.

Case 4: F is $\forall x G(x)$, or $\exists x G(x)$. By induction on $G(\xi^\diamond)$ for each ξ in the universe. ∎

Theorem 4 Let F be a first-order sentence of signature σ and $\mathbf{c}$ *be a list of predicate and function constants. For any interpretation I of σ,* $I \models \text{SM}[F; \mathbf{c}]$ *iff*

- $\langle I, I\rangle \models_{\rm fht} F$*, and*
- *for every interpretation J of σ such that $J <^{\mathbf{c}} I$, we have $\langle J, I\rangle \not\models_{\rm fht} F$.*

Proof. We use Theorem 1 to refer to the reduct-based reformulation and instead show

- I satisfies F, and
- every interpretation J such that $J <^{\mathbf{c}} I$ does not satisfy $(gr_I[F])^{\underline{I}}$

iff

- $\langle I, I \rangle \models_{\text{fht}} F$, and
- for every interpretation J of σ such that $J <^{\mathbf{c}} I$, we have $\langle J, I \rangle \not\models_{\text{fht}} F$.

Clearly, $I \models F$ iff $\langle I, I \rangle \models_{\text{fht}} F$. By Lemma 9, for every interpretation J such that $J <^{\mathbf{c}} I$, we have $J \not\models (gr_I[F])^{\underline{I}}$ iff $\langle J, I \rangle \not\models_{\text{fht}} F$. ∎

C.4 Proof of Theorem 5

Lemma 10
Let F be a $\mathbf{c}$-plain sentence of signature σ, let I, K be total interpretations of σ, and let J be a partial interpretation of σ such that

- $J \prec^{\mathbf{c}} I$ and $K <^{\mathbf{c}} I$;
- $p^J = p^K$ for every predicate constant;
- $f^J(\boldsymbol{\xi}) = u$ iff $f^K(\boldsymbol{\xi}) \neq f^I(\boldsymbol{\xi})$ for every function constant f and every $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f.

We have $K \models gr_I[F]^{\underline{I}}$ iff $J \models_{\text{p}} gr_I[F]^{\underline{I}}$.

Proof.

Case 1: F is an atomic sentence of the form $p(\mathbf{t})$. Since F is $\mathbf{c}$-plain, $\mathbf{t}$ contains no constants from $\mathbf{c}$, and by the assumption $J \prec^{\mathbf{c}} I$ and $K <^{\mathbf{c}} I$, we have $\mathbf{t}^J = \mathbf{t}^K = \mathbf{t}^I$. Since J and K agree on p, the claim holds.

Case 2: F is an atomic sentence of the form $f(\mathbf{t}) = t_1$.

- *Subcase 1:* $I \not\models f(\mathbf{t}) = t_1$. Then $gr_I[F]^{\underline{I}}$ is $\bot$, so the claim holds.
- *Subcase 2:* $I \models f(\mathbf{t}) = t_1$. Then $gr_I[F]^{\underline{I}}$ is $f(\mathbf{t}) = t_1$. Further, from the assumption that F is $\mathbf{c}$-plain, $\mathbf{t}$ and t_1 contain no constants from $\mathbf{c}$, and by the assumptions that $J \prec^{\mathbf{c}} I$, $K <^{\mathbf{c}} I$ and that I is total, we have $\mathbf{t}^J = \mathbf{t}^K = \mathbf{t}^I \neq u$ and $t_1^J = t_1^K = t_1^I \neq u$.
  Either $f(\mathbf{t})^J \neq u$ or $f(\mathbf{t})^J = u$. In the first case, since $J \prec^{\mathbf{c}} I$, we have $f(\mathbf{t})^J = f(\mathbf{t})^I$. Also, by the assumption on K, $f(\mathbf{t})^K = f(\mathbf{t})^I$. Consequently, $J \models_{\text{p}} f(\mathbf{t}) = t_1$ and $K \models f(\mathbf{t}) = t_1$.
  In the second case, $J \not\models_{\text{p}} f(\mathbf{t}) = t_1$. Also, by the assumption on K, $f(\mathbf{t})^K \neq f(\mathbf{t})^I = t_1^I = t_1^K$, so $K \not\models f(\mathbf{t}) = t_1$.

The other cases are straightforward. ∎

Recall the definitions: for two classical interpretations I, K of the same signature σ with the same universe and a list $\mathbf{c}$ of distinct predicate and function constants, we write $K <^{\mathbf{c}} I$ if

$$K \text{ and } I \text{ agree on all constants in } \sigma \setminus \mathbf{c}, \tag{C2}$$

$$p^K \subseteq p^I \text{ for all predicates } p \text{ in } \mathbf{c}, \text{ and} \tag{C3}$$

$$K \text{ and } I \text{ do not agree on } \mathbf{c}. \tag{C4}$$

Similarly, for two partial interpretations J and I of the same signature σ over the same universe $|I|$, and a set of constants $\mathbf{c}$, $J \prec^{\mathbf{c}} I$ is equivalent to

$$J \text{ and } I \text{ agree on all constants in } \sigma \setminus \mathbf{c}, \tag{C5}$$

$$p^J \subseteq p^I \text{ for all predicates } p \text{ in } \mathbf{c}, \text{ and} \tag{C6}$$

$$J \text{ and } I \text{ do not agree on } \mathbf{c} \tag{C7}$$

with the additional requirement that

$$\text{for every function constant } f \in \mathbf{c}\text{, and every } \boldsymbol{\xi} \in |I|^n \text{ where } n \text{ is the arity of } f,\ f^I(\boldsymbol{\xi}) = f^J(\boldsymbol{\xi}) \text{ or } f^J(\boldsymbol{\xi}) = u. \tag{C8}$$

If we drop (C7), this is equivalent to $J \preceq^{\mathbf{c}} I$.

Lemma 11
Let F be a $\mathbf{c}$-plain sentence of signature σ, and let I be total interpretation of σ that satisfies $\exists xy(x \neq y)$. There is a partial interpretation J such that $J \prec^{\mathbf{c}} I$ and $J \models_{\mathrm{p}} gr_I[F]^{\underline{I}}$ iff there is a total interpretation K such that $K <^{\mathbf{c}} I$ and $K \models gr_I[F]^{\underline{I}}$.

Proof. *Left-to-right:* Let J be a partial interpretation such that $J \prec^{\mathbf{c}} I$ and $J \models_{\mathrm{p}} gr_I[F]^{\underline{I}}$. We construct the total interpretation K as follows. For each constant d not in $\mathbf{c}$, $d^K = d^J = d^I$. For each predicate constant p in $\mathbf{c}$ and each $\boldsymbol{\xi} \in |I|^n$ where n is the arity of p,

$$p^K(\boldsymbol{\xi}) = p^J(\boldsymbol{\xi})\ ,$$

and, for each function constant f in $\mathbf{c}$ and each $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f,

$$f^K(\boldsymbol{\xi}) = \begin{cases} f^I(\boldsymbol{\xi}) & \text{if } f^J(\boldsymbol{\xi}) \neq u; \\ m(f^I(\boldsymbol{\xi})) & \text{otherwise} \end{cases}$$

where m is a mapping $m : |I| \to |I|$ such that $\forall x(m(x) \neq x)$ (note that such a mapping requires $I \models \exists xy(x \neq y)$).

We now show that $K <^{\mathbf{c}} I$. It is immediate from the assumption $J \prec^{\mathbf{c}} I$ and by definition that (C2) and (C3) hold. Consider the following cases.

- Case 1: For every function constant $f \in \mathbf{c}$ and every $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f, $f^J(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ (note that since I is total, these cannot be u). From (C7), it follows that there is at least one predicate constant p in $\mathbf{c}$ such that $p^J \subset p^I$. However, by the definition of K, $p^K \subset p^I$ and so (C4) holds.
- Case 2: There is some function constant $f \in \mathbf{c}$ and some $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f such that $f^J(\boldsymbol{\xi}) \neq f^I(\boldsymbol{\xi})$. From (C8), it follows that $f^J(\boldsymbol{\xi}) = u$ and thus by the definition of K, $f^K(\boldsymbol{\xi}) = m(f^I(\boldsymbol{\xi})) \neq f^I(\boldsymbol{\xi})$ and so (C4) holds.

By Lemma 10, the fact $K \models gr_I[F]^{\underline{I}}$ follows from the assumption $J \models_{\mathrm{p}} gr_I[F]^{\underline{I}}$.

Right-to-left: Let K be a total interpretation such that $K <^{\mathbf{c}} I$ and $K \models gr_I[F]^{\underline{I}}$. We construct the partial interpretation J as follows. For each constant d not in $\mathbf{c}$, $d^K = d^J = d^I$. For each predicate constant p in $\mathbf{c}$ and each $\boldsymbol{\xi} \in |I|^n$ where n is the arity of p,

$$p^J(\boldsymbol{\xi}) = p^K(\boldsymbol{\xi})\ ,$$

and, for each function constant f in $\mathbf{c}$ and each $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f,

$$f^J(\boldsymbol{\xi}) = \begin{cases} f^I(\boldsymbol{\xi}) & \text{if } f^K(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi}); \\ u & \text{otherwise.} \end{cases}$$

We now show that $J \prec^{\mathbf{c}} I$. It is immediate from the assumption that $K <^{\mathbf{c}} I$ and by definition that (C5) and (C6) hold. Consider the following cases.

- Case 1: For every function constant $f \in \mathbf{c}$ and every $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f, $f^K(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$. By the definition of J, $f^J(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ and so (C8) holds. Now since

(C4) holds, there is at least one predicate constant p such that $p^K \subset p^I$. However, by the definition of J, $p^J \subset p^I$ and so (C7) holds.

- Case 2: There is some function constant $f \in \mathbf{c}$ and some $\boldsymbol{\xi} \in |I|^n$ where n is the arity of f such that $f^K(\boldsymbol{\xi}) \neq f^I(\boldsymbol{\xi})$. For such a function f, by the definition of J, it must be that $f^J(\boldsymbol{\xi}) = u$. For other functions $f' \in \mathbf{c}$ such that $(f')^K(\boldsymbol{\xi}') = (f')^I(\boldsymbol{\xi}')$ for every $\boldsymbol{\xi}'$, as in Case 1, we conclude $(f')^J(\boldsymbol{\xi}) = (f')^I(\boldsymbol{\xi})$. Consequently, (C8) and (C7) both hold.

By Lemma 10, the fact $J \models_{\rm p} gr_I[F]^{\underline{I}}$ follows from the assumption $K \models gr_I[F]^{\underline{I}}$.

Theorem 5 *For any* $\mathbf{c}$*-plain sentence* F *of signature* σ*, any list* $\mathbf{c}$ *of intensional constants, and any total interpretation* I *of* σ *satisfying* $\exists xy(x \neq y)$, $I \models \mathrm{SM}[F; \mathbf{c}]$ *iff* $I \models_{\rm p} \mathrm{CBL}[F; \mathbf{c}]$.

Proof. We use Theorem 1 and Theorem 2 to refer to the grounding and reduct based definitions rather than the second-order logic based definitions. The claim follows from Lemma 11. ∎

C.5 Proof of Theorem 7 and Corollary 1

Lemma 12
For any partial interpretation I and any atomic sentence $p(t_1, \dots, t_k)$ and $f(t_1, \dots, t_{k-1}) = t_k$,

(a) $I \models_{\rm p} p(t_1, \dots, t_k)$ iff

$$I \models_{\rm p} \exists x_{n_1} \dots x_{n_j}(p(t_1, \dots, t_k)'' \wedge t_{n_1} = x_{n_1} \wedge \dots \wedge t_{n_j} = x_{n_j})$$

where $\{n_1, \dots, n_j\} \subseteq \{1, \dots, k\}$ and $p(t_1, \dots, t_k)''$ is obtained from $p(t_1, \dots, t_k)$ by replacing each t_{n_i} in $p(t_1, \dots, t_k)$ with x_{n_i}.

(b) $I \models_{\rm p} f(t_1, \dots, t_{k-1}) = t_k$ iff

$$I \models_{\rm p} \exists x_{n_1} \dots x_{n_j}((f(t_1, \dots, t_{k-1}) = t_k)'' \wedge t_{n_1} = x_{n_1} \wedge \dots \wedge t_{n_j} = x_{n_j})$$

where $\{n_1, \dots, n_j\} \subseteq \{1, \dots, k\}$ and $(f(t_1, \dots, t_{k-1}) = t_k)''$ is obtained from $f(t_1, \dots, t_{k-1}) = t_k$ by replacing each t_{n_i} in $f(t_1, \dots, t_{k-1}) = t_k$ with x_{n_i}.

Proof. Consider the following cases.

Case 1: $t_i^I = u$ for some $i \in \{n_1, \dots, n_j\}$. Clearly, $I \not\models_{\rm p} p(t_1, \dots, t_k)$ and $I \not\models_{\rm p} f(t_1, \dots, t_{k-1}) = t_k$. It is also the case that $I \not\models_{\rm p} t_i = \boldsymbol{\xi}^\diamond$ for any $\boldsymbol{\xi} \in |I|$ so we have

$$I \not\models_{\rm p} \exists x_{n_1} \dots x_{n_j}(p(t_1, \dots, t_k)'' \wedge t_{n_1} = x_{n_1} \wedge \dots \wedge t_{n_j} = x_{n_j}) \tag{C9}$$

and

$$I \not\models_{\rm p} \exists x_{n_1} \dots x_{n_j}((f(t_1, \dots, t_{k-1}) = t_k)'' \wedge t_{n_1} = x_{n_1} \wedge \dots \wedge t_{n_j} = x_{n_j})\,. \tag{C10}$$

Case 2: $t_i^I = u$ for some $i \in \{1, \dots, k\} \setminus \{n_1, \dots, n_j\}$. Clearly, $I \not\models_{\rm p} p(t_1, \dots, t_k)$ and $I \not\models_{\rm p} f(t_1, \dots, t_{k-1}) = t_k$. Also, since t_i remains in $p(t_1, \dots, t_k)''$ and $(f(t_1, \dots, t_k) = t)''$, we have $I \not\models_{\rm p} p(t_1, \dots, t_k)''$ and $I \not\models_{\rm p} (f(t_1, \dots, t_k) = t)''$, from which (C9) and (C10) follow.

Case 3: $t_i^I \neq u$ for all $i \in \{1, \dots, k\}$. Condition (a) clearly holds because it coincides with classical equivalence. For Condition (b), consider two subcases:

- *Subcase 1:* $f(t_1, \dots, t_{k-1})^I \neq u$. Clearly, Condition (b) coincides with classical equivalence.

- *Subcase 2:* $f(t_1, \ldots, t_{k-1})^I = u$. Clearly, $I \not\models_{\rm p} f(t_1, \ldots, t_{k-1}) = t_k$. Now in

$$\exists x_{n_1} \ldots x_{n_j}((f(t_1, \ldots, t_{k-1}) = t_k)'' \wedge t_{n_1} = x_{n_1} \wedge \cdots \wedge t_{n_j} = x_{n_j}),$$

  there is only one set of values for $x_{n_1} \ldots x_{n_j}$ that satisfies the last j conjunctive terms—when x_{n_i} is mapped to $t^I_{n_i}$. However, for this set of values, $((f(t_1, \ldots, t_{k-1}))'')^I = f(t_1, \ldots, t_{k-1})^I = u$ (where $(f(t_1, \ldots, t_{k-1}))''$ is obtained from $f(t_1, \ldots, t_{k-1})$ by replacing each t_{n_i} with x_{n_i}) so (C10) holds.

∎

Lemma 13
Given a sentence F, a set of constants $\mathbf{c}$, and a partial interpretation I, we have $I \models_{\rm p} F$ iff $I \models_{\rm p} UF_{\mathbf{c}}(F)$.

Proof. The proof is by induction on the number of unfolding that needs to be done. More precisely, for any formula F, we define $NU_{\mathbf{c}}(F)$ ("Needed Unfolding") as follows.

- $NU_{\mathbf{c}}(p(t_1, \ldots, t_k)) =$
  $\begin{cases} 0 & \text{if } p(t_1, \ldots, t_k) \text{ is } \mathbf{c}\text{-plain;} \\ \max(NU_{\mathbf{c}}(t_1 = x), \ldots, NU_{\mathbf{c}}(t_k = x)) + 1 & \text{otherwise.} \end{cases}$
- $NU_{\mathbf{c}}(f(t_1, \ldots, t_{k-1}) = t_k) =$
  $\begin{cases} 0 & \text{if } f(t_1, \ldots, t_{k-1}) = t_k \text{ is } \mathbf{c}\text{-plain;} \\ \max(NU_{\mathbf{c}}(t_1 = x), \ldots, NU_{\mathbf{c}}(t_k = x)) + 1 & \text{otherwise.} \end{cases}$
- $NU_{\mathbf{c}}(G \odot H) = \max(NU_{\mathbf{c}}(G), NU_{\mathbf{c}}(H)) + 1$, where $\odot \in \{\wedge, \vee, \rightarrow\}$.
- $NU_{\mathbf{c}}(QxG) = NU_{\mathbf{c}}(G) + 1$, where $Q \in \{\forall, \exists\}$.

Case 1: F is a $\mathbf{c}$-plain atomic sentence. F is identical to $UF_{\mathbf{c}}(F)$ so the claim holds.

Case 2: F is $p(\mathbf{t})$ where $\mathbf{t}$ contains at least one constant from $\mathbf{c}$. Let $t_{n_1} \ldots t_{n_j}$ be the j terms in $\mathbf{t}$ containing at least one constant from $\mathbf{c}$. Now $UF_{\mathbf{c}}(F)$ is $\exists x_{n_1} \ldots x_{n_j}(p(t_1, \ldots, t_k)'' \wedge UF_{\mathbf{c}}(t_{n_1} = x_{n_1}) \wedge \cdots \wedge UF_{\mathbf{c}}(t_{n_j} = x_{n_j}))$ where $p(t_1, \ldots, t_k)''$ is obtained from $p(t_1, \ldots, t_k)$ by replacing each t_{n_i} in $p(t_1, \ldots, t_k)$ with x_{n_i}. Since $NU_{\mathbf{c}}(F) > NU_{\mathbf{c}}(t_{n_i} = \boldsymbol{\xi}^\diamond)$ for each $\boldsymbol{\xi} \in |I|$ and each $i \in \{1, \ldots, j\}$, by I.H. on $t_{n_i} = \boldsymbol{\xi}^\diamond$, $UF_{\mathbf{c}}(t_{n_i} = x_{n_i})$ can be replaced by $t_{n_i} = x_{n_i}$ so that $I \models_{\rm p} UF_{\mathbf{c}}(F)$ iff $I \models_{\rm p} \exists x_{n_1} \ldots x_{n_j}(p(t_1, \ldots, t_k)'' \wedge t_{n_1} = x_{n_1} \wedge \cdots \wedge t_{n_j} = x_{n_j})$. By Lemma 12 the latter is equivalent to $I \models_{\rm p} F$.

Case 3: F is $f(\mathbf{t}) = t_1$ where at least one of $\mathbf{t}$ and t_1 contain at least one constant from $\mathbf{c}$. Let $t_{n_1} \ldots t_{n_j}$ be the j terms in $\mathbf{t}$ and t_1 containing at least one constant from $\mathbf{c}$. Now $UF_{\mathbf{c}}(F)$ is $\exists x_{n_1} \ldots x_{n_j}((f(\mathbf{t}) = t_1)'' \wedge UF_{\mathbf{c}}(t_{n_1} = x_{n_1}) \wedge \cdots \wedge UF_{\mathbf{c}}(t_{n_j} = x_{n_j}))$, where $(f(\mathbf{t}) = t_1)''$ is obtained from $f(\mathbf{t}) = t_1$ by replacing each t_{n_i} in $f(\mathbf{t}) = t_1$ with x_{n_i}. Since $NU_{\mathbf{c}}(F) > NU_{\mathbf{c}}(t_{n_i} = \boldsymbol{\xi}^\diamond)$ for each $\boldsymbol{\xi} \in |I|$ and each $i \in \{1, \ldots, j\}$, by I.H. on $t_{n_i} = \boldsymbol{\xi}^\diamond$, $UF_{\mathbf{c}}(t_{n_i} = x_{n_i})$ can be replaced by $t_{n_i} = x_{n_i}$ so that $I \models_{\rm p} UF_{\mathbf{c}}(F)$ iff $I \models_{\rm p} \exists x_{n_1} \ldots x_{n_j}((f(\mathbf{t}) = t_1)'' \wedge t_{n_1} = x_{n_1} \wedge \cdots \wedge t_{n_j} = x_{n_j})$. By Lemma 12 the latter is equivalent to $I \models_{\rm p} F$.

Case 4: F is $G \odot H$ for $\odot \in \{\wedge, \vee, \rightarrow\}$. By I.H. on G and H.

Case 5: F is $QxF(x)$ for $Q \in \{\forall, \exists\}$. By I.H. on $F(\boldsymbol{\xi}^\diamond)$ for each $\boldsymbol{\xi} \in |I|$. ∎

Theorem 7 For any sentence F, any list $\mathbf{c}$ of constants, and any partial interpretation I, we have $I \models_{\rm p} \mathrm{CBL}[F; \mathbf{c}]$ iff $I \models_{\rm p} \mathrm{CBL}[UF_{\mathbf{c}}(F); \mathbf{c}]$.

Proof. By definition, CBL$[F;\mathbf{c}]$ is

$$F \wedge \neg\exists\widehat{\mathbf{c}}(\widehat{\mathbf{c}}\prec\mathbf{c} \wedge F^{\dagger}(\widehat{\mathbf{c}}))$$

and CBL$[UF_{\mathbf{c}}(F);\mathbf{c}]$ is by definition

$$UF_{\mathbf{c}}(F) \wedge \neg\exists\widehat{\mathbf{c}}(\widehat{\mathbf{c}}\prec\mathbf{c} \wedge (UF_{\mathbf{c}}(F))^{\dagger}(\widehat{\mathbf{c}})).$$

Now, for any partial interpretation I of signature $\sigma \supseteq \mathbf{c}$, by Lemma 13, $I \models_{\mathrm{p}} F$ iff $I \models_{\mathrm{p}} UF_{\mathbf{c}}(F)$. It is sufficient to show that, for any partial interpretation J, $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \mathbf{d} \prec \mathbf{c} \wedge F^{\dagger}(\mathbf{d})$ iff $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} \mathbf{d} \prec \mathbf{c} \wedge (UF_{\mathbf{c}}(F))^{\dagger}(\mathbf{d})$.

Case 1: F is an atomic sentence. $F^{\dagger}(\mathbf{d})$ is $F(\mathbf{d})$, and $UF_{\mathbf{c}}(F)^{\dagger}(\mathbf{d})$ is $UF_{\mathbf{c}}(F)(\mathbf{d})$. $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} F(\mathbf{d})$ iff $J \models_{\mathrm{p}} F$. Similarly, $J^{\mathbf{c}}_{\mathbf{d}} \cup I \models_{\mathrm{p}} UF_{\mathbf{c}}(F)(\mathbf{d})$ iff $J \models_{\mathrm{p}} UF_{\mathbf{c}}(F)$. By Lemma 12, $J \models_{\mathrm{p}} F$ iff $J \models_{\mathrm{p}} UF_{\mathbf{c}}(F)$, so the claim follows.

Case 2: F is $G \odot H$ for $\odot \in \{\wedge, \vee\}$. By induction on G and H.

Case 3: F is $G \rightarrow H$. $F^{\dagger}(\mathbf{d})$ is $(G^{\dagger}(\mathbf{d}) \rightarrow H^{\dagger}(\mathbf{d})) \wedge (G \rightarrow H)$ and $(UF_{\mathbf{c}}(F))^{\dagger}(\mathbf{d})$ is $(UF_{\mathbf{c}}(G))^{\dagger}(\mathbf{d}) \rightarrow (UF_{\mathbf{c}}(H))^{\dagger}(\mathbf{d})) \wedge (UF_{\mathbf{c}}(G) \rightarrow UF_{\mathbf{c}}(H))$. The equivalence between the first conjunctive terms (under partial satisfaction) is by I.H. on G and H, and the equivalence between the second conjunctive terms (under partial satisfaction) is by Lemma 13.

Case 4: F is $QxG(x)$ for $Q \in \{\forall, \exists\}$. By I.H. on $G(\boldsymbol{\xi}^{\diamond})$ for each $\boldsymbol{\xi} \in |I|$. ∎

Corollary 1 For any sentence F, any list $\mathbf{c}$ of constants, and any total interpretation I satisfying $\exists xy(x \neq y)$, we have $I \models_{\mathrm{p}}$ CBL$[F;\mathbf{c}]$ iff $I \models_{\mathrm{p}}$ CBL$[UF_{\mathbf{c}}(F);\mathbf{c}]$ iff $I \models$ SM$[UF_{\mathbf{c}}(F);\mathbf{c}]$.

Proof. The equivalence between the first and the second conditions is by Theorem 7. The equivalence between the second and the third conditions is by Theorem 5 since $UF_{\mathbf{c}}(F)$ is $\mathbf{c}$-plain. ∎

C.6 Proof of Theorem 6

Theorem 6 For any head-$\mathbf{c}$-plain sentence F of signature σ that is tight on $\mathbf{c}$, and any total interpretation I of σ satisfying $\exists xy(x \neq y)$, $I \models$ SM$[F;\mathbf{c}]$ iff $I \models_{\mathrm{p}}$ CBL$[F;\mathbf{c}]$.

Proof. We first note that since F is head-$\mathbf{c}$-plain and tight on $\mathbf{c}$, we can transform this into Clark normal form that is still tight on $\mathbf{c}$, so we can assume that F is already turned into this form.

By Corollary 1, $I \models_{\mathrm{p}}$ CBL$[F;\mathbf{c}]$ iff $I \models$ SM$[UF_{\mathbf{c}}(F);\mathbf{c}]$, so it remains to check that $I \models$ SM$[UF_{\mathbf{c}}(F);\mathbf{c}]$ iff $I \models$ SM$[F;\mathbf{c}]$.

It is easy to check that the completion of $UF_{\mathbf{c}}(F)$ relative to $\mathbf{c}$ is equivalent to the completion of F relative to $\mathbf{c}$. By Theorem 2 from (Bartholomew and Lee 2013), we conclude that SM$[UF_{\mathbf{c}}(F);\mathbf{c}]$ is equivalent to SM$[F;\mathbf{c}]$. ∎

C.7 Proof of Theorem 8 , Corollary 2 , and Corollary 3

Theorem 8 For any f-plain sentence F and any partial interpretation I, if

$$I \models_{\mathrm{p}} \forall \mathbf{x}y(p(\mathbf{x},y) \leftrightarrow f(\mathbf{x})=y) \tag{C11}$$

then $I \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$ *iff* $I \models_{\rm p} \text{CBL}[F^f_p;\ p, \mathbf{c}]$.

Proof. For any partial interpretation I of signature $\sigma \supseteq \{f, p, \mathbf{c}\}$ satisfying (C11), it is clear that $I \models_{\rm p} F$ iff $I \models_{\rm p} F^f_p$ since F^f_p is simply the result of replacing all $f(\mathbf{x}) = y$ with $p(\mathbf{x}, y)$. Thus it is sufficient to show that

$$I \models_{\rm p} \exists \widehat{f}\widehat{\mathbf{c}}\Big((\widehat{f}, \widehat{\mathbf{c}}) \prec (f, \mathbf{c}) \wedge F^\dagger(\widehat{f}, \widehat{\mathbf{c}})\Big) \text{ iff } I \models_{\rm p} \exists \widehat{p}\widehat{\mathbf{c}}\Big((\widehat{p}, \widehat{\mathbf{c}}) \prec (p, \mathbf{c}) \wedge (F^f_p)^\dagger(\widehat{p}, \widehat{\mathbf{c}})\Big).$$

Left-to-right: Assume $I \models_{\rm p} \exists \widehat{f}\widehat{\mathbf{c}}((\widehat{f}, \widehat{\mathbf{c}}) \prec (f, \mathbf{c}) \wedge F^\dagger(\widehat{f}, \widehat{\mathbf{c}}))$. We wish to show that $I \models_{\rm p} \exists \widehat{p}\widehat{\mathbf{c}}((\widehat{p}, \widehat{\mathbf{c}}) \prec (p, \mathbf{c}) \wedge (F^f_p)^\dagger(\widehat{p}, \widehat{\mathbf{c}}))$. That is, take any function g of the same arity as f and any list of predicate and function constants $\mathbf{d}$ that is similar to $\mathbf{c}$. For any partial interpretation J of signature σ, $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I$ is an interpretation of the extended signature $\sigma' = \sigma \cup \{g, q, \mathbf{d}\}$. We assume

$$J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g, \mathbf{d}) \prec (f, \mathbf{c}) \wedge F^\dagger(g, \mathbf{d})$$

and wish to show that there is a predicate q of the same arity as p such that

$$J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q, \mathbf{d}) \prec (p, \mathbf{c}) \wedge (F^f_p)^\dagger(q, \mathbf{d}).$$

We define the new predicate q in terms of g as follows:

$$q^{J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I}(\boldsymbol{\xi}, \xi') = \begin{cases} \text{TRUE} & \text{if } g^{J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I}(\boldsymbol{\xi}) = \xi' \ ; \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

We first show if $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g, \mathbf{d}) \prec (f, \mathbf{c})$ then $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q, \mathbf{d}) \prec (p, \mathbf{c})$.

Case 1: $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \not\models_{\rm p} g \prec f$. Since we assume $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g, \mathbf{d}) \prec (f, \mathbf{c})$, it follows that

$$J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \ \models_{\rm p} \ g = f \ , \tag{C12}$$

and $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} \mathbf{d} \prec \mathbf{c}$. From (C11), (C12), and the definition of q, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} q = p$. Consequently, $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q, \mathbf{d}) \prec (p, \mathbf{c})$.

Case 2: $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} g \prec f$. From (C11), $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} g \prec f$, and the definition of q, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} q \prec p$. Since we assume $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g, \mathbf{d}) \prec (f, \mathbf{c})$, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} \mathbf{d} \preceq \mathbf{c}$. Consequently, $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q, \mathbf{d}) \prec (p, \mathbf{c})$.

We now show that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (F^f_p)^\dagger(q, \mathbf{d})$ by proving $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} F^\dagger(g, \mathbf{d})$ iff $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (F^f_p)^\dagger(q, \mathbf{d})$.

Case 1: F is an f-plain atomic sentence of the form $p(\mathbf{t})$, or $t_1 = t_2$ such that t_1 does not contain f. The claim is obvious since F^f_p is exactly F and so $(F^f_p)^\dagger(q, \mathbf{d})$ is exactly $F^\dagger(g, \mathbf{d})$.

Case 2: F is an f-plain atomic sentence of the form $f(\mathbf{t}) = t_1$. Then $F^\dagger(g, \mathbf{d})$ is $g(\mathbf{t}') = t'_1$, where $\mathbf{t}'$ and t'_1 are obtained from $\mathbf{t}$ and t_1 by replacing the members of $\mathbf{c}$ with the corresponding members of $\mathbf{d}$. F^f_p is $p(\mathbf{t}, t_1)$, and $(F^f_p)^\dagger(q, \mathbf{d})$ is $q(\mathbf{t}', t'_1)$. From the definition of q, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} g(\mathbf{t}') = t'_1 \leftrightarrow q(\mathbf{t}', t'_1)$.

Case 3: F is $G \odot H$ where $\odot \in \{\wedge, \vee, \rightarrow\}$. By I.H. on G and H.

Case 4: F is $QxG(x)$ where $Q \in \{\forall, \exists\}$. By I.H. on $G(\xi^\diamond)$ for each $\xi \in |I|$.

Right-to-left: Assume $I \models_{\rm p} \exists \widehat{p}\widehat{\mathbf{c}}((\widehat{p},\widehat{\mathbf{c}}) \prec (p,\mathbf{c}) \wedge (F^f_p)^\dagger(\widehat{p},\widehat{\mathbf{c}}))$. We wish to show that $I \models_{\rm p} \exists(\widehat{f},\widehat{\mathbf{c}})((\widehat{f},\widehat{\mathbf{c}}) \prec (f,\mathbf{c}) \wedge F^\dagger(\widehat{f},\widehat{\mathbf{c}}))$. That is, take any predicate q of the same arity as p and any list of predicates and functions $\mathbf{d}$ that is similar to $\mathbf{c}$. As before, let J be a partial interpretation of σ, and $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I$ is an interpretation of the extended signature $\sigma' = \sigma \cup \{g, q, \mathbf{d}\}$. We assume

$$J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q,\mathbf{d}) \prec (p,\mathbf{c}) \wedge (F^f_p)^\dagger(q,\mathbf{d})$$

and wish to show that there is a function g of the same arity as f such that

$$J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g,\mathbf{d}) \prec (f,\mathbf{c}) \wedge F^\dagger(g,\mathbf{d}).$$

We define $g^{J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I}$ in terms of q as follows:

$$g^{J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I}(\boldsymbol{\xi}) = \begin{cases} f^{J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I}(\boldsymbol{\xi}) & \text{if } q^{J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I}(\boldsymbol{\xi}, f^{J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I}(\boldsymbol{\xi})) = \text{TRUE}\ ; \\ u & \text{otherwise.} \end{cases}$$

We first show that if $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q,\mathbf{d}) \prec (p,\mathbf{c})$ then $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g,\mathbf{d}) \prec (f,\mathbf{c})$.

Case 1: $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} q = p$. Since we assume $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q,\mathbf{d}) \prec (p,\mathbf{c})$, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} \mathbf{d} \prec \mathbf{c}$. From (C11), $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} q = p$, and by the definition of g, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} g = f$. Consequently, $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g,\mathbf{d}) \prec (f,\mathbf{c})$.

Case 2: $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} \neg(q = p)$. Since we assume $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q,\mathbf{d}) \prec (p,\mathbf{c})$, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} q \preceq p$ and so we have

$$J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} q \prec p\ . \tag{C13}$$

From (C11), (C13), and the definition of g, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} g \prec f$. Also from the assumption that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (q,\mathbf{d}) \prec (p,\mathbf{c})$, it follows that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} \mathbf{d} \preceq \mathbf{c}$. Consequently, $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (g,\mathbf{d}) \prec (f,\mathbf{c})$.

We show that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} F^\dagger(g,\mathbf{d})$ by proving that $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} F^\dagger(g,\mathbf{d})$ iff $J^{(f,\mathbf{c})}_{(g,\mathbf{d})} \cup I \models_{\rm p} (F^f_p)^\dagger(q,\mathbf{d})$. The proof is similar to the one above, and is omitted.

Corollary 2 Let F be an f-plain sentence. (a) For any partial interpretation I of the signature of F, $I \models_{\rm p} \mathrm{CBL}[F;\ f,\mathbf{c}]$ iff $I^f_p \models_{\rm p} \mathrm{CBL}[F^f_p \wedge UC_p;\ p,\mathbf{c}]$. (b) For any partial interpretation J of the signature of F^f_p, $J \models_{\rm p} \mathrm{CBL}[F^f_p \wedge UC_p;\ p,\mathbf{c}]$ iff $J = I^f_p$ for some partial interpretation I such that $I \models_{\rm p} \mathrm{CBL}[F;\ f,\mathbf{c}]$.

Proof. For two partial interpretations I of signature σ_1 and J of signature σ_2 with the same universe, by $I \cup J$ we denote the partial interpretation of signature $\sigma_1 \cup \sigma_2$ that interprets all constants occurring only in σ_1 in the same way as I does and similarly for σ_2 and J. For constants appearing in both σ_1 and σ_2, I must interpret these the same as J does, in which case $I \cup J$ also interprets the constants in this way.

Part (a), Left-to-right: Assume $I \models_{\rm p} \mathrm{CBL}[F;\ f,\mathbf{c}]$. By the definition of I^f_p, $I \cup I^f_p \models_{\rm p}$ (C11). Thus by Theorem 8, $I \cup I^f_p \models_{\rm p} \mathrm{CBL}[F;\ f,\mathbf{c}] \leftrightarrow \mathrm{CBL}[F^f_p;\ p,\mathbf{c}]$. Since we assume $I \models_{\rm p} \mathrm{CBL}[F;\ f,\mathbf{c}]$, it is the case that $I \cup I^f_p \models_{\rm p} \mathrm{CBL}[F;\ f,\mathbf{c}]$ and thus it must be the case that $I \cup I^f_p \models_{\rm p} \mathrm{CBL}[F^f_p;\ p,\mathbf{c}]$.

Further, (C11) entails UC_p, so $I \cup I_p^f \models_{\rm p} UC_p$. Since the signature of I does not contain p, we conclude $I_p^f \models_{\rm p} \text{CBL}[F_p^f;\ p, \mathbf{c}] \wedge UC_p$ and since UC_p is comprised of constraints, $I_p^f \models_{\rm p} \text{CBL}[F_p^f \wedge UC_p;\ p, \mathbf{c}]$.[3]

Part (a), Right-to-left: Assume $I_p^f \models_{\rm p} \text{CBL}[F_p^f \wedge UC_p;\ p, \mathbf{c}]$. By the definition of I_p^f, $I \cup I_p^f \models_{\rm p}$ (C11). Thus by Theorem 8, $I \cup I_p^f \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}] \leftrightarrow \text{CBL}[F_p^f;\ p, \mathbf{c}]$. From the assumption, we have $I_p^f \models_{\rm p} \text{CBL}[F_p^f;\ p, \mathbf{c}]$, and further $I \cup I_p^f \models_{\rm p} \text{CBL}[F_p^f;\ p, \mathbf{c}]$. Consequently, $I \cup I_p^f \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$, and since the signature of I_p^f does not contain f, we conclude $I \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$.

Part (b), Left-to-right: Assume $J \models_{\rm p} \text{CBL}[F_p^f \wedge UC_p;\ p, \mathbf{c}]$. Let $I = J_f^p$ where J_f^p denotes the partial interpretation of the signature of F obtained from J by replacing the set p^J with the function f such that $f^I(\xi_1, \dots, \xi_k) = \xi_{k+1}$ for all tuples $\langle \xi_1, \dots, \xi_k, \xi_{k+1} \rangle$ in p^J. This is a valid definition of a function since we assume $J \models_{\rm p} \text{CBL}[F_p^f \wedge UC_p;\ p, \mathbf{c}]$, from which it follows that $J \models_{\rm p} UC_p$. Clearly, $J = I_p^f$ so it only remains to be shown that $I \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$. By the definition of J_f^p, $I \cup J \models_{\rm p}$ (C11). Thus by Theorem 8, $I \cup J \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}] \leftrightarrow \text{CBL}[F_p^f;\ p, \mathbf{c}]$. From the assumption, we have $J \models_{\rm p} \text{CBL}[F_p^f;\ p, \mathbf{c}]$, and further $I \cup J \models_{\rm p} \text{CBL}[F_p^f;\ p, \mathbf{c}]$. Consequently, $I \cup J \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$, and since the signature of J does not contain f, we conclude $I \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$.

Part (b), Right-to-left: Take any I such that $J = I_p^f$ and $I \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$. By the definition of $J = I_p^f$, $I \cup J \models_{\rm p}$ (C11). Thus by Theorem 8, $I \cup J \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}] \leftrightarrow \text{CBL}[F_p^f;\ p, \mathbf{c}]$. Since we assume $I \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$, it is the case that $I \cup J \models_{\rm p} \text{CBL}[F;\ f, \mathbf{c}]$ and thus it must be the case that $I \cup J \models_{\rm p} \text{CBL}[F_p^f;\ p, \mathbf{c}]$. Further, (C11) entails UC_p, so $I \cup J \models_{\rm p} UC_p$. Since the signature of I does not contain p, we conclude $J \models_{\rm p} \text{CBL}[F_p^f;\ p, \mathbf{c}] \wedge UC_p$ and since UC_p is comprised of constraints, $J \models_{\rm p} \text{CBL}[F_p^f \wedge UC_p;\ p, \mathbf{c}]$. ∎

Corollary 3 Let $\mathbf{c}$ be a set of intensional constants consisting of intensional function constants $\mathbf{f}$ and intensional predicate constants, and let F be an $\mathbf{c}$-plain sentence. (a) For any total interpretation I of the signature of F, $I \models_{\rm p} \text{CBL}[F; \mathbf{c}]$ iff $I_{\mathbf{p}}^{\mathbf{f}} \models \text{SM}[F_{\mathbf{p}}^{\mathbf{f}} \wedge UC_{\mathbf{p}}; \mathbf{c}_{\mathbf{p}}^{\mathbf{f}}]$. (b) For any total interpretation J of the signature of $F_{\mathbf{p}}^{\mathbf{f}}$, $J \models \text{SM}[F_{\mathbf{p}}^{\mathbf{f}} \wedge UC_{\mathbf{p}}; \mathbf{c}_{\mathbf{p}}^{\mathbf{f}}]$ iff $J = I_{\mathbf{p}}^{\mathbf{f}}$ for some total interpretation I such that $I \models_{\rm p} \text{CBL}[F; \mathbf{c}]$.

Proof. (a) First, by multiple applications of Corollary 2, it follows that for any total interpretation I of the signature of F, $I \models_{\rm p} \text{CBL}[F; \mathbf{c}]$ iff $I_{\mathbf{p}}^{\mathbf{f}} \models_{\rm p} \text{CBL}[F_{\mathbf{p}}^{\mathbf{f}} \wedge UC_{\mathbf{p}}; \mathbf{c}_{\mathbf{p}}^{\mathbf{f}}]$. Then the statement follows from Theorem 5 since $F_{\mathbf{p}}^{\mathbf{f}} \wedge UC_{\mathbf{p}}$ is $\mathbf{c}$-plain.

The proof of (b) is similar. ∎

C.8 Proof of Theorem 9

Given a program Π, by Π^{FOL} we denote the *FOL* representation of Π.

[3] The last step is justified by the theorem on constraints, similar to Theorem 3 from (Ferraris et al. 2011), which we omit here.

Lemma 14
Consider a signature σ and a set of constants $\mathbf{c}$. Given an ASP{f} program Π of signature σ not containing strong negation,

(a) For any partial interpretation I of signature σ that maps every constant in $\sigma \setminus \mathbf{c}$ to itself, there is a consistent set S of seed literals such that $I \models_{\mathrm{p}} \Pi^{FOL}$ iff $S \models_{\mathrm{b}} \Pi$.
(b) For any consistent set of seed literals S, there is a partial interpretation I such that $I \models_{\mathrm{p}} \Pi^{FOL}$ iff $S \models_{\mathrm{b}} \Pi$.

Proof. *Part (a):* Given a partial interpretation I, let S be the set $\{f(\mathbf{v}) = w : f(\mathbf{v})^I = w\} \cup \{p(\mathbf{v}) : p(\mathbf{v})^I = \text{TRUE}\}$. We note that this is a consistent set of seed literals since a partial interpretation maps $f(\mathbf{v})$ to at most one object constant.

We also note that by the definition of S, for any atomic sentence A, we have $I \models_{\mathrm{p}} A$ iff $S \models_{\mathrm{b}} A$. Now, consider any rule r from Π. $I \models_{\mathrm{p}} r^{FOL}$ iff $I \models_{\mathrm{p}} head(r)^{FOL}$ or $I \not\models_{\mathrm{p}} body(r)^{FOL}$. By the previous observation, this is equivalent to $S \models_{\mathrm{b}} head(r)$ or $S \not\models_{\mathrm{b}} body(r)$ since $body(r)$ is a conjunction of atomic formulas. This is precisely the definition of $S \models_{\mathrm{b}} r$.

Part (b): Given a consistent set of seed literals S, let I be the partial interpretation defined as follows:

- for every object constant $v \in \sigma \setminus \mathbf{c}$, we have $v^I = v$.
- for every predicate constant $p \in \mathbf{c}$ and every list of object constants $\mathbf{v}$, we have $p(\mathbf{v})^I =$ TRUE iff $p(\mathbf{v}) \in S$.
- for every function constant $f \in \mathbf{c}$ and every list of object constants $\mathbf{v}$, we have $f(\mathbf{v})^I = u$ if S does not mention $f(\mathbf{v})$, and $f(\mathbf{v})^I = w$ if $f(\mathbf{v}) = w$ is in S.

We note that the last bullet is well-defined since S is a consistent set of seed literals so that there cannot be two distinct object constants a and b such that $f(\mathbf{v}) = a \in S$ and $f(\mathbf{v}) = b \in S$.

We also note that by the definition of I, for any atomic sentence A, we have $I \models_{\mathrm{p}} A$ iff $S \models_{\mathrm{b}} A$. Now, consider any rule r from Π. $S \models_{\mathrm{b}} r$ iff $S \models_{\mathrm{b}} head(r)$ or $S \not\models_{\mathrm{b}} body(r)$. By the previous observation, this is equivalent to $I \models_{\mathrm{p}} head(r)^{FOL}$ or $I \not\models_{\mathrm{p}} body(r)^{FOL}$ since $body(r)$ is a conjunction of atomic formulas. This is precisely the definition of $I \models_{\mathrm{p}} r^{FOL}$. ∎

The proof of Lemma 14 tells us that a consistent set of seed literals can be identified with a partial interpretation.

Lemma 15
For consistents sets of seed literals J and I of the same signature, J is a proper subset of I iff $J \prec^{\mathbf{c}} I$ (as defined in Section 2.3.2) when we view them as partial interpretations.

Proof. We first note that since consistent sets of literals map every object constant in $\sigma \setminus \mathbf{c}$ to itself, the partial interpretation view does the same which corresponds to the first condition for $J \prec^{\mathbf{c}} I$. The second condition of $J \prec^{\mathbf{c}} I$ is $p^J \subseteq p^I$ for all predicate constants in $\mathbf{c}$, which corresponds exactly to the predicate part of J being a subset of the predicate part of I. Finally, the third condition of $J \prec^{\mathbf{c}} I$ is $f^J(\boldsymbol{\xi}) = u$ or $f^J(\boldsymbol{\xi}) = f^I(\boldsymbol{\xi})$ corresponds to the function part of J being a subset of the function part of I since we identify a partial interpretation mapping an element to u to the absence of that element in the set. ∎

Theorem 9 For any ASP{f} program Π with intensional constants $\mathbf{c}$ and any consistent set I of seed literals, if Π has no strong negation, then I is a Balduccini answer set of Π iff $I \models_{\mathrm{p}}$ $\text{CBL}[\Pi; \mathbf{c}]$.

Proof. By definition and by using the equivalent reformulation presented and justified in Lemma 15 and Lemma 14, I is a Balduccini answer set of a program Π iff $I \models_{\rm p} \Pi$ and for every partial interpretation J such that $J \prec^{\bf c} I$, we have $J \not\models_{\rm p} \Pi^I$. This is equivalent to the reduct reformulation of the Cabalar semantics. Further, this is equivalent to $I \models_{\rm p} \text{CBL}[\Pi^{FOL}; {\bf c}]$ by Theorem 2. ∎

C.9 Proof of Theorem 10

Theorem 10 *For any ASP$\{f\}$ program Π with intensional constants ${\bf c}$ and any consistent set I of seed literals, I is a Balduccini answer set of Π iff I is a Balduccini answer set of $\Pi^{\#}$.*

Proof. First, we show that $I \models_{\rm b} \sim(f = g)$ iff $I \models_{\rm b} (f = f) \wedge (g = g) \wedge \neg(f = g)$.

Left-to-right: Assume $I \models_{\rm b} \sim(f = g)$. By definition, I contains both $f = c_1$ and $g = c_2$ for some object constants c_1 and c_2 such that $c_1 \neq c_2$. Clearly, each of $I \models f = f$, $I \models g = g$ and $I \not\models f = g$ holds.

Right-to-left: $I \models_{\rm b} (f = f) \wedge (g = g) \wedge \neg(f = g)$. Since $I \models_{\rm b} f = f$ and $I \models g = g$, it follows that I contains $f = c_1$ and I contains $f = c_2$ for some c_1 and c_2. Further, since $I \models \neg(f = g)$, it must be that $c_1 \neq c_2$, from which the claim follows.

From this it is not difficult to check that Π^I is equivalent to $(\Pi^{\#})^I$ under partial satisfaction, from which the claim follows. ∎

C.10 Proof of Theorem 11

Theorem 11 *For any sentence F in Clark normal form that is tight on ${\bf c}$ and any total interpretation I, if $I \models \exists xy(x \neq y)$, then $I \models_{\rm p} \text{CBL}[F; {\bf c}]$ iff $I \models \text{SM}[F; {\bf c}]$ iff I is a model of the completion of F relative to ${\bf c}$.*

Proof. By Theorem 2 from (Bartholomew and Lee 2013), I is a model of the completion of F relative to ${\bf c}$ iff $I \models \text{SM}[F; {\bf c}]$. Since a formula in Clark normal form that is tight on ${\bf c}$ is also head-${\bf c}$-plain and is tight on ${\bf c}$, $I \models \text{SM}[F; {\bf c}]$ iff $I \models_{\rm p} \text{CBL}[F; {\bf c}]$ by Theorem 6. ∎

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