
FUZZY PROPOSITIONAL FORMULAS UNDER THE STABLE MODEL SEMANTICS

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Abstract

We define a stable model semantics for fuzzy propositional formulas, which generalizes both fuzzy propositional logic and the stable model semantics of classical propositional formulas. The syntax of the language is the same as the syntax of fuzzy propositional logic, but its semantics distinguishes stable models from non-stable models. The generality of the language allows for highly configurable nonmonotonic reasoning for dynamic domains involving graded truth degrees. We show that several properties of Boolean stable models are naturally extended to this many-valued setting, and discuss how it is related to other approaches to combining fuzzy logic and the stable model semantics.

1 Introduction

Answer set programming (ASP) [Lifschitz, 2008] is a widely applied declarative programming paradigm for the design and implementation of knowledge intensive applications. One of the attractive features of ASP is its capability to represent the nonmonotonic aspect of knowledge. However, its mathematical basis, the stable

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model semantics, is restricted to Boolean values, and is too rigid to represent imprecise and vague information. Fuzzy logic, as a form of many-valued logic, can handle such information by interpreting propositions with graded truth degrees, often as real numbers in the interval $[0, 1]$. The availability of various fuzzy operators gives the user great flexibility in combining the truth degrees. However, the semantics of fuzzy logic is monotonic and is not flexible enough to handle default reasoning as in answer set programming.

Both the stable model semantics and fuzzy logic are generalizations of classical propositional logic in different ways. While they do not subsume each other, it is clear that many real-world problems require both their strengths. This has led to the body of work on combining fuzzy logic and the stable model semantics, known as fuzzy answer set programming (FASP) (e.g., [Lukasiewicz, 2006; Janssen *et al.*, 2012; Vojtás, 2001; Damásio and Pereira, 2001b; Medina *et al.*, 2001; Damásio and Pereira, 2001a; Van Nieuwenborgh *et al.*, 2007; Madrid and Ojeda-Aciego, 2008]). However, most works consider simple rule forms and do not allow connectives nested arbitrarily as in fuzzy logic. An exception is fuzzy equilibrium logic [Schockaert *et al.*, 2012], which applies to arbitrary propositional formulas even including strong negation. However, its definition is highly complex.

Unlike the existing works on fuzzy answer set semantics, in this paper, we extend the general stable model semantics from [Ferraris, 2005; Ferraris *et al.*, 2011] to fuzzy propositional formulas. The syntax of the language is the same as the syntax of fuzzy propositional logic. The semantics, on the other hand, defines a condition which distinguishes *stable* models from non-stable models. The language is a proper generalization of both fuzzy propositional logic and classical propositional formulas under the stable model semantics, and turns out to be essentially equivalent to fuzzy equilibrium logic, but is much simpler. Unlike the interval-based semantics in fuzzy equilibrium logic, the proposed semantics is based on the notions of a reduct and a minimal model, familiar from the usual way stable models are defined, and thus provides a simpler, alternative characterization of fuzzy equilibrium logic. In fact, in the absence of strong negation, a fuzzy equilibrium model always assigns an interval of the form $[v, 1]$ to each atom, which can be simply identified with a single value v under the stable model semantics. Further we show that strong negation can be eliminated from a formula in favor of new atoms, extending the well-known result in answer set programming. So our simple semantics fully captures fuzzy equilibrium logic.

Also, there is a significant body of work based on the general stable model semantics, such as the splitting theorem [Ferraris *et al.*, 2009], the theorem on loop formulas [Ferraris *et al.*, 2006], and the concept of aggregates [Lee *et al.*, 2008]. The simplicity of our semantics would allow for easily extending those results to the

many-valued setting, as can be seen from some examples in this paper.

Another contribution of this paper is to show how reasoning about dynamic systems in ASP can be extended to fuzzy ASP. It is well known that actions and their effects on the states of the world can be conveniently represented by answer set programs [Lifschitz and Turner, 1999; Lifschitz, 2002]. On the contrary, to the best of our knowledge, the work on fuzzy ASP has not addressed how the result can be extended to many-valued setting, and most applications discussed so far are limited to static domains only. As a motivating example, consider modeling dynamics of *trust* in social networks. People trust each other in different degrees under some normal assumptions. If person A trusts person B , then A tends to trust person C whom B trusts, to a degree which is positively correlated to the degree to which A trusts B and the degree to which B trusts C . By default, the trust degrees would not change, but may decrease when some conflict arises between the people. Modeling such a domain requires expressing defaults involving fuzzy truth values. We demonstrate how the proposed language can achieve this by taking advantage of its generality over the existing approaches to fuzzy ASP. Thus the generalization is not simply a pure theoretical pursuit, but has practical uses in conveniently modeling defaults involving fuzzy truth values in dynamic domains.

The paper is organized as follows. Section 2 reviews the syntax and the semantics of fuzzy propositional logic, as well as the stable model semantics of classical propositional formulas. Section 3 presents the main definition of a stable model of a fuzzy propositional formula along with several examples. Section 4 presents a generalized definition based on partial degrees of satisfaction and its reduction to the special case, as well as an alternative, second-order logic style definition. Section 5 tells us how the Boolean stable model semantics can be viewed as a special case of the fuzzy stable model semantics, and Section 6 formally compares the fuzzy stable model semantics with normal FASP programs and fuzzy equilibrium logic. Section 7 shows that some well-known properties of the Boolean stable model semantics can be naturally extended to our fuzzy stable model semantics. Section 8 discusses other related work, followed by the conclusion in Section 9. The complete proofs are given in the appendix.

This paper is a significantly extended version of the papers [Lee and Wang, 2014a; Lee and Wang, 2014b]. Instead of the second-order logic style definition used there, we present a new, reduct-based definition as the main definition, which is simpler and more aligned with the standard definition of a stable model. Further, this paper shows that a generalization of the stable model semantics that allows partial degrees of satisfaction can be reduced to the version that allows only the two-valued concept of satisfaction.

2 Preliminaries

2.1 Review: Fuzzy Logic

A fuzzy propositional signature σ is a set of symbols called *fuzzy atoms*. In addition, we assume the presence of a set CONJ of fuzzy conjunction symbols, a set DISJ of fuzzy disjunction symbols, a set NEG of fuzzy negation symbols, and a set IMPL of fuzzy implication symbols.

A *fuzzy (propositional) formula* (of σ) is defined recursively as follows.

- every fuzzy atom $p \in \sigma$ is a fuzzy formula;
- every numeric constant c , where c is a real number in $[0, 1]$, is a fuzzy formula;
- if F is a fuzzy formula, then $\neg F$ is a fuzzy formula, where $\neg \in \text{NEG}$;
- if F and G are fuzzy formulas, then $F \otimes G$, $F \oplus G$, and $F \rightarrow G$ are fuzzy formulas, where $\otimes \in \text{CONJ}$, $\oplus \in \text{DISJ}$, and $\rightarrow \in \text{IMPL}$.

The models of a fuzzy formula are defined as follows [Hajek, 1998]. The *fuzzy truth values* are the real numbers in the range $[0, 1]$. A *fuzzy interpretation* I of σ is a mapping from σ into $[0, 1]$.

The fuzzy operators are functions mapping one or a pair of fuzzy truth values into a truth value. Among the operators, \neg denotes a function from $[0, 1]$ into $[0, 1]$; \otimes , \oplus , and \rightarrow denote functions from $[0, 1] \times [0, 1]$ into $[0, 1]$. The actual mapping performed by each operator can be defined in many different ways, but all of them satisfy the following conditions, which imply that the operators are generalizations of the corresponding classical propositional connectives:¹

- a fuzzy negation \neg is decreasing, and satisfies $\neg(0) = 1$ and $\neg(1) = 0$;
- a fuzzy conjunction \otimes is increasing, commutative, associative, and $\otimes(1, x) = x$ for all $x \in [0, 1]$;
- a fuzzy disjunction \oplus is increasing, commutative, associative, and $\oplus(0, x) = x$ for all $x \in [0, 1]$;
- a fuzzy implication \rightarrow is decreasing in its first argument and increasing in its second argument; and $\rightarrow(1, x) = x$ and $\rightarrow(0, 0) = 1$ for all $x \in [0, 1]$.

Figure 1 lists some specific fuzzy operators that we use in this paper.

¹We say that a function f of arity n is *increasing in its i -th argument* ($1 \leq i \leq n$) if $f(\text{arg}_1, \dots, \text{arg}_i, \dots, \text{arg}_n) \leq f(\text{arg}_1, \dots, \text{arg}'_i, \dots, \text{arg}_n)$ whenever $\text{arg}_i \leq \text{arg}'_i$; f is said to be *increasing* if it is increasing in all its arguments. The definition of *decreasing* is similarly defined.

Symbol	Name	Definition
\otimes_l	Łukasiewicz t-norm	$\otimes_l(x, y) = \max(x + y - 1, 0)$
\oplus_l	Łukasiewicz t-conorm	$\oplus_l(x, y) = \min(x + y, 1)$
\otimes_m	Gödel t-norm	$\otimes_m(x, y) = \min(x, y)$
\oplus_m	Gödel t-conorm	$\oplus_m(x, y) = \max(x, y)$
\otimes_p	product t-norm	$\otimes_p(x, y) = x \cdot y$
\oplus_p	product t-conorm	$\oplus_p(x, y) = x + y - x \cdot y$
\neg_s	standard negator	$\neg_s(x) = 1 - x$
\rightarrow_r	R-implicator induced by \otimes_m	$\rightarrow_r(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
\rightarrow_s	S-implicator induced by \otimes_m	$\rightarrow_s(x, y) = \max(1 - x, y)$
\rightarrow_l	Implicator induced by \otimes_l	$\rightarrow_l(x, y) = \min(1 - x + y, 1)$

Figure 1: Some t-norms, t-conorms, negator, and implicators

The *truth value* of a fuzzy propositional formula F under I , denoted $v_I(F)$, is defined recursively as follows:

- for any atom $p \in \sigma$, $v_I(p) = I(p)$;
- for any numeric constant c , $v_I(c) = c$;
- $v_I(\neg F) = \neg(v_I(F))$;
- $v_I(F \odot G) = \odot(v_I(F), v_I(G))$ ($\odot \in \{\otimes, \oplus, \rightarrow\}$).

(For simplicity, we identify the symbols for the fuzzy operators with the truth value functions represented by them.)

Definition 1. We say that a fuzzy interpretation I satisfies a fuzzy formula F if $v_I(F) = 1$, and denote it by $I \models F$. We call such I a fuzzy model of F .

We say that two formulas F and G are equivalent if $v_I(F) = v_I(G)$ for all interpretations I , and denote it by $F \Leftrightarrow G$.

2.2 Review: Stable Models of Classical Propositional Formulas

A *propositional signature* is a set of symbols called *atoms*. A propositional formula is defined recursively using atoms and the following set of primitive propositional connectives: \perp , \top , \neg , \wedge , \vee , \rightarrow .

An *interpretation* of a propositional signature is a function from the signature into $\{\text{FALSE}, \text{TRUE}\}$. We identify an interpretation with the set of atoms that are true in it.

A *model* of a propositional formula is an interpretation that *satisfies* the formula. According to [Ferraris, 2005], the models are divided into stable models and non-stable models as follows. The *reduct* F^X of a propositional formula F relative to a set X of atoms is the formula obtained from F by replacing every maximal subformula that is not satisfied by X with \perp . Set X is called a *stable model* of F if X is a minimal set of atoms satisfying F^X .

Alternatively, the reduct F^X can be defined recursively as follows:

Definition 2. • When F is an atom or \perp or \top , $F^X = \begin{cases} F & \text{if } X \models F; \\ \perp & \text{otherwise;} \end{cases}$

• $(\neg F)^X = \begin{cases} \perp & \text{if } X \models F; \\ \top & \text{otherwise;} \end{cases}$

• For $\odot \in \{\wedge, \vee, \rightarrow\}$,

$$(F \odot G)^X = \begin{cases} F^X \odot G^X & \text{if } X \models F \odot G; \\ \perp & \text{otherwise.} \end{cases}$$

In the next section, we extend this definition to cover fuzzy propositional formulas.

3 Definition and Examples

3.1 Reduct-based Definition

Let σ be a fuzzy propositional signature, F a fuzzy propositional formula of σ , and I an interpretation of σ .

Definition 3. The (fuzzy) reduct of F relative to I , denoted F^I , is defined recursively as follows:

- For any fuzzy atom or numeric constant F , $F^I = F$;
- $(\neg F)^I = v_I(\neg F)$;
- $(F \odot G)^I = (F^I \odot G^I) \otimes_m v_I(F \odot G)$, where $\odot \in \{\otimes, \oplus, \rightarrow\}$.

Compare this definition and Definition 2. They are structurally similar, but also have subtle differences. One of them is that in the case of binary connectives \odot , instead of distinguishing the cases between when X satisfies the formula or not as in Definition 2, Definition 3 keeps a conjunction of $F^I \odot G^I$ and $v_I(F \odot G)$.²

Another difference is that in the case of an atom, Definition 3 is a bit simpler as it does not distinguish between the two cases. It turns out that the same clause can be applied to Definition 2 (i.e., $F^X = F$ when F is an atom regardless of $X \models F$), still yielding an equivalent definition of a Boolean stable model. So this difference is not essential.

For any two fuzzy interpretations J and I of signature σ and any subset \mathbf{p} of σ , we write $J \leq^{\mathbf{p}} I$ if

- $J(p) = I(p)$ for each fuzzy atom not in \mathbf{p} , and
- $J(p) \leq I(p)$ for each fuzzy atom p in \mathbf{p} .

We write $J <^{\mathbf{p}} I$ if $J \leq^{\mathbf{p}} I$ and $J \neq I$. We may simply write it as $J < I$ when $\mathbf{p} = \sigma$.

Definition 4. *We say that an interpretation I is a (fuzzy) stable model of F relative to \mathbf{p} (denoted $I \models \text{SM}[F; \mathbf{p}]$) if*

- $I \models F$, and
- there is no interpretation J such that $J <^{\mathbf{p}} I$ and J satisfies F^I .

If \mathbf{p} is the same as the underlying signature, we may simply write $\text{SM}[F; \mathbf{p}]$ as $\text{SM}[F]$ and drop the clause “relative to \mathbf{p} .”

We call an interpretation J such that $J <^{\mathbf{p}} I$ and J satisfies F^I as a witness to dispute the stability of I (for F relative to \mathbf{p}). In other words, a model of F is stable if it has no witness to dispute its stability for F .

Clearly, when \mathbf{p} is empty, Definition 4 reduces to the definition of a fuzzy model in Definition 1 because simply there is no interpretation J such that $J <^{\emptyset} I$.

The definition of a reduct can be simplified in the cases of \otimes and \oplus , which are increasing in both arguments:

- $(F \otimes G)^I = (F^I \otimes G^I)$; $(F \oplus G)^I = (F^I \oplus G^I)$.

This is due to the following proposition, which can be proved by induction.

²In fact, a straightforward modification of the second subcases in Definition 2 by replacing \perp with some fixed truth values does not work for the fuzzy stable model semantics.

Proposition 1. *For any interpretations I and J such that $J \leq^{\mathbf{P}} I$, it holds that*

$$v_J(F^I) \leq v_I(F).$$

In the following we will assume this simplified form of a reduct.

Also, we may view $\neg F$ as shorthand for some fuzzy implication $F \rightarrow 0$. For instance, what we called the standard negator can be derived from the residual implicator \rightarrow_l induced by Łukasiewicz t-norm, defined as $\rightarrow_l(x, y) = \min(1-x+y, 1)$. In view of Proposition 1, for $J \leq^{\mathbf{P}} I$,

$$\begin{aligned} v_J((F \rightarrow_l 0)^I) &= v_J((F^I \rightarrow_l 0) \otimes_m v_I(F \rightarrow_l 0)) \\ &= v_J(v_I(F \rightarrow_l 0)) = v_I(F \rightarrow_l 0) = v_I(\neg_s F). \end{aligned}$$

Thus the second clause of Definition 3 can be viewed as a special case of the third clause.

Example 1. *Consider the fuzzy formula $F_1 = \neg_s q \rightarrow_r p$ and the interpretation $I = \{(p, x), (q, 1-x)\}$, where $x \in [0, 1]$. $I \models F_1$ because*

$$v_I(\neg_s q \rightarrow_r p) = \rightarrow_r(x, x) = 1.$$

The reduct F_1^I is

$$((\neg_s q)^I \rightarrow_r p) \otimes_m 1 \Leftrightarrow (v_I(\neg_s q) \rightarrow_r p) \Leftrightarrow x \rightarrow_r p$$

I can be a stable model only if $x = 1$. Otherwise, $\{(p, x), (q, 0)\}$ is a witness to dispute the stability of I .

*On the other hand, for $F_2 = (\neg_s q \rightarrow_r p) \otimes_m (\neg_s p \rightarrow_r q)$, for any value of $x \in [0, 1]$, I is a stable model.*³

$$\begin{aligned} F_2^I &= (\neg_s q \rightarrow_r p)^I \otimes_m (\neg_s p \rightarrow_r q)^I \\ &\Leftrightarrow (v_I(\neg_s q) \rightarrow_r p) \otimes_m 1 \otimes_m (v_I(\neg_s p) \rightarrow_r q) \otimes_m 1 \\ &\Leftrightarrow (x \rightarrow_r p) \otimes_m ((1-x) \rightarrow_r q). \end{aligned}$$

No interpretation J such that $J <^{\{p,q\}} I$ satisfies F_2^I .

Example 2. • $F_1 = p \rightarrow_r p$ is a tautology (i.e., every interpretation is a model of the formula), but not all models are stable. First, $I_1 = \{(p, 0)\}$ is a stable model. The reduct $F_1^{I_1}$ is

$$(p^I \rightarrow_r p^I) \otimes_m 1 \Leftrightarrow p \rightarrow_r p \Leftrightarrow 1$$

³Strictly speaking, $(1-x)$ in the reduct should be understood as the value of the arithmetic function applied to the arguments.

and obviously there is no witness to dispute the stability of I .

No other interpretation $I_2 = \{(p, x)\}$ where $x > 0$ is a stable model. The reduct $F_1^{I_2}$ is again equivalent to 1, but I_1 serves as a witness to dispute the stability of I_2 .

- $F_2 = \neg_s \neg_s p \rightarrow_r p$ is equivalent to F_1 , but their stable models are different. Any $I = \{(p, x)\}$, where $x \in [0, 1]$, is a stable model of F_2 . The reduct F_2^I is

$$\begin{aligned} ((\neg_s \neg_s p)^I \rightarrow_r p^I) \otimes_m 1 &\Leftrightarrow v_I(\neg_s \neg_s p) \rightarrow_r p \\ &\Leftrightarrow x \rightarrow_r p. \end{aligned}$$

No interpretation J such that $J <^{\{p\}} I$ satisfies F_2^I .

- Let F_3 be $\neg_s p \oplus_l p$. Any $I = \{(p, x)\}$, where $x \in [0, 1]$, is a stable model of F_3 . The reduct F_3^I is

$$(\neg_s p)^I \oplus_l p^I \Leftrightarrow v_I(\neg_s p) \oplus_l p \Leftrightarrow (1 - x) \oplus_l p.$$

No interpretation J such that $J <^p I$ satisfies the reduct.

The following proposition extends a well-known fact about the relationship between a formula and its reduct in terms of satisfaction.

Proposition 2. *A (fuzzy) interpretation I satisfies a formula F if and only if I satisfies F^I .*

For any fuzzy formula F , any interpretation I and any set \mathbf{p} of atoms, we say that I is a *minimal* model of F relative to \mathbf{p} if I satisfies F and there is no interpretation J such that $J <^{\mathbf{p}} I$ and J satisfies F . Using this notion, Proposition 2 tells us that Definition 4 can be reformulated as follows.

Corollary 1. *An interpretation I is a (fuzzy) stable model of F relative to \mathbf{p} iff I is a minimal model of F^I relative to \mathbf{p} .*

This reformulation relies on the fact that \otimes_m is intended in the third bullet of Definition 3 rather than an arbitrary fuzzy conjunction because we want the truth value of the “conjunction” of $F^I \rightarrow G^I$ and $F \rightarrow G$ to be either the truth value of $F^I \rightarrow G^I$ or the truth value $F \rightarrow G$. Conjunctions that do not have this property lead to unintuitive behaviors, such as violating Proposition 2. As an example, consider the formula

$$F = 0.6 \rightarrow_r (1 \rightarrow_r p)$$

and the interpretation

$$I = \{(p, 0.6)\}.$$

Clearly, I satisfies F . According to Definition 3,

$$\begin{aligned} F^I &= (0.6 \rightarrow_r (1 \rightarrow_r p)^I) \otimes_m v_I(0.6 \rightarrow_r (1 \rightarrow_r p)) \\ &= (0.6 \rightarrow_r ((1 \rightarrow_r p) \otimes_m v_I(1 \rightarrow_r p))) \otimes_m 1 \\ &= 0.6 \rightarrow_r ((1 \rightarrow_r p) \otimes_m 0.6) \end{aligned}$$

and

$$\begin{aligned} v_I(F^I) &= (0.6 \rightarrow_r ((1 \rightarrow_r 0.6) \otimes_m 0.6)) \\ &= 0.6 \rightarrow_r 0.6 = 1 \end{aligned}$$

So I satisfies F^I as well. However, if we replace \otimes_m by \otimes_l in the third bullet of Definition 3, we get

$$\begin{aligned} v_I(F^I) &= 0.6 \rightarrow_r ((1 \rightarrow_r 0.6) \otimes_l 0.6) \\ &= 0.6 \rightarrow_r 0.2 = 0.2 < 1, \end{aligned}$$

which indicates that I does not satisfy F^I . Therefore, I is a fuzzy stable model of F according to Definition 4, but it is not even a model of the reduct F^I if \otimes_l were used in place of \otimes_m .

Similarly, it can be checked that the same issue remains if we use \otimes_p in place of \otimes_m .

The following example illustrates how the commonsense law of inertia involving fuzzy truth values can be represented.

Example 3. Let σ be $\{p_0, np_0, p_1, np_1\}$ and let F be $F_1 \otimes_m F_2$, where F_1 represents that p_1 and np_1 are complementary, i.e., the sum of their truth values is 1:⁴

$$F_1 = \neg_s(p_1 \otimes_l np_1) \otimes_m \neg_s \neg_s(p_1 \oplus_l np_1).$$

⁴This is similar to the formulas used under the Boolean stable model semantics to express that two Boolean atoms p_1 and np_1 take complementary values, i.e.,

$$\neg(p_1 \wedge np_1) \wedge \neg\neg(p_1 \vee np_1)$$

F_2 represents that by default p_1 has the truth value of p_0 , and np_1 has the truth value of np_0 :⁵

$$F_2 = ((p_0 \otimes_m \neg_s \neg_s p_1) \rightarrow_r p_1) \otimes_m ((np_0 \otimes_m \neg_s \neg_s np_1) \rightarrow_r np_1).$$

One can check that the interpretation

$$I_1 = \{(p_0, x), (np_0, 1 - x), (p_1, x), (np_1, 1 - x)\}$$

(x is any value in $[0, 1]$) is a stable model of F relative to $\{p_1, np_1\}$: F^{I_1} is equivalent to

$$(p_0 \otimes_m x \rightarrow_r p_1) \otimes_m (np_0 \otimes_m (1 - x) \rightarrow_r np_1).$$

No interpretation J such that $J <^{\{p_1, np_1\}} I_1$ satisfies F^{I_1} .

The interpretation

$$I_2 = \{(p_0, x), (np_0, 1 - x), (p_1, y), (np_1, 1 - y)\},$$

where $y > x$, is not a stable model of F . The reduct F^{I_2} is equivalent to

$$(p_0 \otimes_m y \rightarrow_r p_1) \otimes_m (np_0 \otimes_m (1 - y) \rightarrow_r np_1),$$

and the interpretation $\{(p_0, x), (np_0, 1 - x), (p_1, x), (np_1, 1 - y)\}$ serves as a witness to dispute the stability of I_2 .

Similarly, when $y < x$, we can check that I_2 is not a stable model of F relative to $\{p_1, np_1\}$.

On the other hand, if we conjoin F with $y \rightarrow_r p_1$ and $(1 - y) \rightarrow_r np_1$ to yield $F \otimes_m (y \rightarrow_r p_1) \otimes_m ((1 - y) \rightarrow_r np_1)$, then the default behavior is overridden, and I_2 is a stable model of $F \otimes_m (y \rightarrow_r p_1) \otimes_m ((1 - y) \rightarrow_r np_1)$ relative to $\{p_1, np_1\}$.⁶

This behavior is useful in expressing the commonsense law of inertia involving fuzzy values. Suppose p_0 represents the value of fluent p at time 0, and p_1 represents the value at time 1. Then F states that, “by default, the fluent retains the previous value.” The default value is overridden if there is an action that sets p to a different value.

This way of representing the commonsense law of inertia is a straightforward extension of the solution in ASP.

⁵This is similar to the rules used in ASP to express the commonsense law of inertia, e.g.,

$$p_0 \wedge \neg \neg p_1 \rightarrow p_1$$

⁶One may wonder why the part $(1 - y) \rightarrow_r np_1$ is also needed. It can be checked that if we drop the part $(1 - y) \rightarrow_r np_1$ and have y less than x , then I_2 (with $y < x$) is not a stable model of $F \otimes_m (y \rightarrow_r p_1)$ relative to $\{p_1, np_1\}$ because $J = \{(p_0, x), (np_0, 1 - x), (p_1, y), (np_1, 1 - x)\}$ disputes the stability of I_2 .

Example 4. *The trust example in the introduction can be formalized in the fuzzy stable model semantics as follows. Below x, y, z are schematic variables ranging over people, and t is a schematic variable ranging over time steps. $\text{Trust}(x, y, t)$ is a fuzzy atom representing that “ x trusts y at time t .” Similarly, $\text{Distrust}(x, y, t)$ is a fuzzy atom representing that “ x distrusts y at time t .”*

The trust relation is reflexive:

$$F_1 = \text{Trust}(x, x, t).$$

The trust and distrust degrees are complementary, i.e., their sum is 1 (similar to F_1 in Example 3):

$$\begin{aligned} F_2 &= \neg_s(\text{Trust}(x, y, t) \otimes_l \text{Distrust}(x, y, t)), \\ F_3 &= \neg_s \neg_s(\text{Trust}(x, y, t) \oplus_l \text{Distrust}(x, y, t)). \end{aligned}$$

Initially, if x trusts y to degree d_1 and y trusts z to degree d_2 , then we assume x trusts z to degree $d_1 \times d_2$; furthermore the initial distrust degree is 1 minus the initial trust degree.

$$\begin{aligned} F_4 &= \text{Trust}(x, y, 0) \otimes_p \text{Trust}(y, z, 0) \rightarrow_r \text{Trust}(x, z, 0), \\ F_5 &= \neg_s \text{Trust}(x, y, 0) \rightarrow_r \text{Distrust}(x, y, 0). \end{aligned}$$

The inertia assumption (similar to F_2 in Example 3):

$$\begin{aligned} F_6 &= \text{Trust}(x, y, t) \otimes_m \neg_s \neg_s \text{Trust}(x, y, t+1) \rightarrow_r \text{Trust}(x, y, t+1), \\ F_7 &= \text{Distrust}(x, y, t) \otimes_m \neg_s \neg_s \text{Distrust}(x, y, t+1) \rightarrow_r \text{Distrust}(x, y, t+1). \end{aligned}$$

A conflict increases the distrust degree by the conflict degree:

$$F_8 = \text{Conflict}(x, y, t) \oplus_l \text{Distrust}(x, y, t) \rightarrow_r \text{Distrust}(x, y, t+1),$$

Let F_{TW} be $F_1 \otimes_m F_2 \otimes_m \cdots \otimes_m F_8$. Suppose we have the formula $F_{\text{Fact}} = \text{Fact}_1 \otimes_m \text{Fact}_2$ that gives the initial trust degree.

$$\begin{aligned} \text{Fact}_1 &= 0.8 \rightarrow_r \text{Trust}(\text{Alice}, \text{Bob}, 0), \\ \text{Fact}_2 &= 0.7 \rightarrow_r \text{Trust}(\text{Bob}, \text{Carol}, 0). \end{aligned}$$

Although there is no fact about how much Alice trusts Carol, any stable model of $F_{TW} \otimes_m F_{\text{Fact}}$ assigns value 0.56 to the atom $\text{Trust}(\text{Alice}, \text{Carol}, 0)$. On the other hand, the stable model assigns value 0 to $\text{Trust}(\text{Alice}, \text{David}, 0)$ due to the closed world assumption under the stable model semantics.

When we conjoin $F_{TW} \otimes F_{Fact}$ with $0.2 \rightarrow_r \text{Conflict}(\text{Alice}, \text{Carol}, 0)$, the stable model of

$$F_{TW} \otimes_m F_{Fact} \otimes_m (0.2 \rightarrow_r \text{Conflict}(\text{Alice}, \text{Carol}, 0))$$

manifests that the trust degree between Alice and Carol decreases to 0.36 at time 1. More generally, if we have more actions that change the trust degree in various ways, by specifying the entire history of actions, we can determine the evolution of the trust distribution among all the participants. Useful decisions can be made based on this information. For example, Alice may decide not to share her personal pictures to those whom she trusts less than degree 0.48.

Note that this example, like Example 3, uses nested connectives, such as $\neg_s \neg_s$, that are not available in the syntax of FASP considered in earlier work, such as [Lukasiewicz, 2006; Janssen *et al.*, 2012].

It is often assumed that, for any fuzzy rule arrow \leftarrow , it holds that $\leftarrow(x, y) = 1$ iff $x \geq y$ [Damásio and Pereira, 2001b]. This condition is required to use \leftarrow to define an immediate consequence operator for a positive program whose least fixpoint coincides with the unique minimal model. (A positive program is a set of rules of the form $a \leftarrow b_1 \otimes \dots \otimes b_n$, where a, b_1, \dots, b_n are atoms.) Notice that \rightarrow_r in Figure 1 satisfies this condition, but \rightarrow_s does not.

4 Generalization and Alternative Definitions

4.1 y -Stable Models

While the fuzzy stable model semantics presented in the previous section allows atoms to have many values, like ASP, it holds on to the two-valued concept of satisfaction, i.e., a formula is either satisfied or not. In a more flexible setting we may allow a formula to be partially satisfied to a certain degree.

First, we generalize the notion of satisfaction to allow partial degrees of satisfaction as in [Van Nieuwenborgh *et al.*, 2007].

Definition 5. For any real number $y \in [0, 1]$, we say that a fuzzy interpretation I y -satisfies a fuzzy formula F if $v_I(F) \geq y$, and denote it by $I \models_y F$. We call I a fuzzy y -model of F .

Using this generalized notion of satisfaction, it is straightforward to generalize the definition of a stable model to incorporate partial degrees of satisfaction.

Definition 6. We say that an interpretation I is a y -stable model of F relative to \mathbf{p} (denoted $I \models_y \text{SM}[F; \mathbf{p}]$) if

- $I \models_y F$ and
- there is no interpretation J such that $J <^{\mathbf{P}} I$ and $J \models_y F^I$.

Fuzzy models and fuzzy stable models as defined in Section 2.1 and Section 3 are fuzzy 1-models and fuzzy 1-stable models according to these generalized definitions.

Example 5. Consider the fuzzy formula $F = \neg_s p \rightarrow_r q$ and the interpretation $I = \{(p, 0), (q, 0.6)\}$. $I \models_{0.6} F$ because $v_I(\neg_s p \rightarrow_r q) = \rightarrow_r(1, 0.6) = 0.6$.

The reduct F^I is

$$((\neg_s p)^I \rightarrow_r q) \otimes_m 1 \Leftrightarrow (v_I(\neg_s p) \rightarrow_r q) \Leftrightarrow 1 \rightarrow_r q$$

Clearly, for any J such that $J <^{\{p,q\}} I$, we observe that $J \not\models_{0.6} F^I$.

On the other hand, the generalized definition is not essential in the sense that it can be reduced to the special case as follows.

Theorem 1. For any fuzzy formula F , an interpretation I is a y -stable model of F relative to \mathbf{p} iff I is a 1-stable model of $y \rightarrow F$ relative to \mathbf{p} as long as the implication \rightarrow satisfies the condition $\rightarrow(x, y) = 1$ iff $y \geq x$.

For example, in accordance with Theorem 1, $\{(p, 0.6)\}$ is a 0.6-stable model of $\neg_s p \rightarrow_r q$, as well as a 1-stable model of $0.6 \rightarrow_r (\neg_s p \rightarrow_r q)$.

4.2 Second-Order Logic Style Definition

In [Ferraris *et al.*, 2011], second-order logic was used to define the stable models of a first-order formula, which is equivalent to the reduct-based definition when the domain is finite. Similarly, but instead of using second-order logic, we can express the same concept using auxiliary atoms that do not belong to the original signature.

Let σ be a set of fuzzy atoms, and let $\mathbf{p} = (p_1, \dots, p_n)$ be a list of distinct atoms belonging to σ , and let $\mathbf{q} = (q_1, \dots, q_n)$ be a list of new, distinct fuzzy atoms not belonging to σ . For two interpretations I and J of σ that agree on all atoms in $\sigma \setminus \mathbf{p}$, $I \cup J_{\mathbf{q}}^{\mathbf{P}}$ denotes the interpretation of $\sigma \cup \mathbf{q}$ such that

- $(I \cup J_{\mathbf{q}}^{\mathbf{P}})(p) = I(p)$ for each atom p in σ , and
- $(I \cup J_{\mathbf{q}}^{\mathbf{P}})(q_i) = J(p_i)$ for each $q_i \in \mathbf{q}$.

For any fuzzy formula F of signature σ , $F^*(\mathbf{q})$ is defined as follows.

- $p_i^* = q_i$ for each $p_i \in \mathbf{p}$;

- $F^* = F$ for any atom $F \notin \mathbf{p}$ or any numeric constant F ;
- $(\neg F)^* = \neg F$;
- $(F \otimes G)^* = F^* \otimes G^*$; $(F \oplus G)^* = F^* \oplus G^*$;
- $(F \rightarrow G)^* = (F^* \rightarrow G^*) \otimes_m (F \rightarrow G)$.

Theorem 2. *A fuzzy interpretation I is a fuzzy stable model of F relative to \mathbf{p} iff*

- $I \models F$, and
- there is no fuzzy interpretation J such that $J <^{\mathbf{p}} I$ and $I \cup J_{\mathbf{q}}^{\mathbf{p}} \models F^*(\mathbf{q})$.

5 Relation to Boolean-Valued Stable Models

The Boolean stable model semantics in Section 2.2 can be embedded into the fuzzy stable model semantics as follows:

For any classical propositional formula F , define F^{fuzzy} to be the fuzzy propositional formula obtained from F by replacing \perp with 0, \top with 1, \neg with \neg_s , \wedge with \otimes_m , \vee with \oplus_m , and \rightarrow with \rightarrow_s . We identify the signature of F^{fuzzy} with the signature of F . Also, for any propositional interpretation I , we define the corresponding fuzzy interpretation I^{fuzzy} as

- $I^{fuzzy}(p) = 1$ if $I(p) = \text{TRUE}$;
- $I^{fuzzy}(p) = 0$ otherwise.

The following theorem tells us that the Boolean-valued stable model semantics can be viewed as a special case of the fuzzy stable model semantics.

Theorem 3. *For any classical propositional formula F and any classical propositional interpretation I , I is a stable model of F relative to \mathbf{p} iff I^{fuzzy} is a stable model of F^{fuzzy} relative to \mathbf{p} .*

Example 6. *Let F be the classical propositional formula $\neg q \rightarrow p$. F has only one stable model $I = \{p\}$. Likewise, as shown in Example 1, $F^{fuzzy} = \neg_s q \rightarrow_s p$ has only one fuzzy stable model $I^{fuzzy} = \{(p, 1), (q, 0)\}$.*

Theorem 3 does not hold for different selections of fuzzy operators, as illustrated by the following example.

Example 7. Let F be the classical propositional formula $p \vee p$. Classical interpretation $I = \{p\}$ is a stable model of F . However, $I^{\text{fuzzy}} = \{(p, 1)\}$ is not a stable model of $F^{\text{fuzzy}} = p \oplus_I p$ because there is $J = \{(p, 0.5)\} < I$ that satisfies $(F^{\text{fuzzy}})^I = p \oplus_I p$.

However, one direction of Theorem 3 still holds for different selections of fuzzy operators.

Theorem 4. For any classical propositional formula F , let F_1^{fuzzy} be the fuzzy formula obtained from F by replacing \perp with 0, \top with 1, \neg with any fuzzy negation symbol, \wedge with any fuzzy conjunction symbol, \vee with any fuzzy disjunction symbol, and \rightarrow with any fuzzy implication symbol. For any classical propositional interpretation I , if I^{fuzzy} is a fuzzy stable model of F_1^{fuzzy} relative to \mathbf{p} , then I is a Boolean stable model of F relative to \mathbf{p} .

6 Relation to Other Approaches to Fuzzy ASP

6.1 Relation to Stable Models of Normal FASP Programs

A normal FASP program [Lukasiewicz, 2006] is a finite set of rules of the form

$$a \leftarrow b_1 \otimes \dots \otimes b_m \otimes \neg b_{m+1} \otimes \dots \otimes \neg b_n,$$

where $n \geq m \geq 0$, a, b_1, \dots, b_n are fuzzy atoms or numeric constants in $[0, 1]$, and \otimes is any fuzzy conjunction. We identify the rule with the fuzzy implication

$$b_1 \otimes \dots \otimes b_m \otimes \neg_s b_{m+1} \otimes \dots \otimes \neg_s b_n \rightarrow_r a,$$

which allows us to say that a fuzzy interpretation I of signature σ satisfies a rule R if $v_I(R) = 1$. I satisfies an FASP program Π if I satisfies every rule in Π .

According to [Lukasiewicz, 2006], an interpretation I is a *fuzzy answer set* of a normal FASP program Π if I satisfies Π , and no interpretation J such that $J <^\sigma I$ satisfies the reduct of Π w.r.t. I , which is the program obtained from Π by replacing each negative literal $\neg b$ with the constant for $1 - I(b)$.

Theorem 5. For any normal FASP program $\Pi = \{r_1, \dots, r_n\}$, let F be the fuzzy formula $r_1 \otimes \dots \otimes r_n$, where \otimes is any fuzzy conjunction. An interpretation I is a fuzzy answer set of Π in the sense of [Lukasiewicz, 2006] if and only if I is a stable model of F .

Example 8. Let Π be the following program

$$\begin{aligned} p &\leftarrow \neg q \\ q &\leftarrow \neg p. \end{aligned}$$

The answer sets of Π according to [Lukasiewicz, 2006] are $\{(p, x), (q, 1 - x)\}$, where x is any value in $[0, 1]$: the corresponding fuzzy formula F is $(\neg_s q \rightarrow_r p) \otimes_m (\neg_s p \rightarrow_r q)$; As we observed in Example 1, its stable models are $\{(p, x), (q, 1 - x)\}$, where x is any real number in $[0, 1]$.

6.2 Relation to Fuzzy Equilibrium Logic

Like the fuzzy stable model semantics introduced in this paper, fuzzy equilibrium logic [Schockaert *et al.*, 2012] generalizes fuzzy ASP programs to arbitrary propositional formulas, but its definition is quite complex as it is based on some complex operations on pairs of intervals and considers strong negation as one of the primitive connectives. Nonetheless we show that fuzzy equilibrium logic is essentially equivalent to the fuzzy stable model semantics where all atoms are subject to minimization.

6.2.1 Review: Fuzzy Equilibrium Logic

We first review the definition of fuzzy equilibrium logic from [Schockaert *et al.*, 2012]. The syntax is the same as the one we reviewed in Section 2.1 except that a new connective \sim (strong negation) may appear in front of atoms.⁷ For any fuzzy propositional signature σ , a (fuzzy N5) *valuation* is a mapping from $\{h, t\} \times \sigma$ to subintervals of $[0, 1]$ such that $V(t, a) \subseteq V(h, a)$ for each atom $a \in \sigma$. For $V(w, a) = [u, v]$, where $w \in \{h, t\}$, we write $V^-(w, a)$ to denote the lower bound u and $V^+(w, a)$ to denote the upper bound v . The *truth value* of a fuzzy formula under V is defined as follows.

- $V(w, c) = [c, c]$ for any numeric constant c ;
- $V(w, \sim a) = [1 - V^+(w, a), 1 - V^-(w, a)]$, where \sim is the symbol for strong negation;
- $V(w, F \otimes G) = [V^-(w, F) \otimes V^-(w, G), V^+(w, F) \otimes V^+(w, G)]$; ⁸
- $V(w, F \oplus G) = [V^-(w, F) \oplus V^-(w, G), V^+(w, F) \oplus V^+(w, G)]$;
- $V(h, \neg F) = [1 - V^-(t, F), 1 - V^-(h, F)]$;
- $V(t, \neg F) = [1 - V^-(t, F), 1 - V^-(t, F)]$;

⁷The definition from [Schockaert *et al.*, 2012] allows strong negation in front of any formulas. We restrict its occurrence only in front of atoms as usual in answer set programs.

⁸For readability, we write the infix notation $(x \odot y)$ in place of $\odot(x, y)$.

- $V(h, F \rightarrow G) = [\min(V^-(h, F) \rightarrow V^-(h, G), V^-(t, F) \rightarrow V^-(t, G)), V^-(h, F) \rightarrow V^+(h, G)];$
- $V(t, F \rightarrow G) = [V^-(t, F) \rightarrow V^-(t, G), V^-(t, F) \rightarrow V^+(t, G)].$

A valuation V is a (fuzzy N5) model of a formula F if $V^-(h, F) = 1$, which implies $V^+(h, F) = V^-(t, F) = V^+(t, F) = 1$. For two valuations V and V' , we say $V' \preceq V$ if $V'(t, a) = V(t, a)$ and $V(h, a) \subseteq V'(h, a)$ for all atoms a . We say $V' \prec V$ if $V' \preceq V$ and $V' \neq V$. We say that a model V of F is *h-minimal* if there is no model V' of F such that $V' \prec V$. An h-minimal fuzzy N5 model V of F is a *fuzzy equilibrium model* of F if $V(h, a) = V(t, a)$ for all atoms a .

6.2.2 In the Absence of Strong Negation

We first establish the correspondence between fuzzy stable models and fuzzy equilibrium models in the absence of strong negation. As in [Schockaert *et al.*, 2012], we assume that the fuzzy negation \neg is \neg_s .

Notice that a fuzzy equilibrium model assigns an interval of values to each atom, rather than a single value as in fuzzy stable models. This accounts for the complexity of the definition of a fuzzy model. However, it turns out that in the absence of strong negation, all upper bounds assigned by a fuzzy equilibrium model are 1.

Lemma 1. *Given a formula F containing no strong negation, any equilibrium model V of F satisfies $V^+(h, a) = V^+(t, a) = 1$ for all atoms a .*

Therefore, in the absence of strong negation, any equilibrium model can be identified with a fuzzy interpretation as follows. For any valuation V , we define a fuzzy interpretation l_V as $\mathsf{l}_V(p) = V^-(h, p)$ for each atom $p \in \sigma$.

The following theorem asserts that there is a 1-1 correspondence between fuzzy equilibrium models and fuzzy stable models.

Theorem 6. *Let F be a fuzzy propositional formula of σ that contains no strong negation.*

- (a) *A valuation V of σ is a fuzzy equilibrium model of F iff $V^-(h, p) = V^-(t, p)$, $V^+(h, p) = V^+(t, p) = 1$ for all atoms p in σ and l_V is a stable model of F relative to σ .*
- (b) *An interpretation I of σ is a stable model of F relative to σ iff $I = \mathsf{l}_V$ for some fuzzy equilibrium model V of F .*

6.2.3 Allowing Strong Negation

In this section we extend the relationship between fuzzy equilibrium logic and our stable model semantics by allowing strong negation. This is done by simulating strong negation by new atoms in our semantics.

Let σ denote the signature. For a fuzzy formula F over σ that may contain strong negation, define F' over $\sigma \cup \{np \mid p \in \sigma\}$ as the formula obtained from F by replacing every strongly negated atom $\sim p$ with a new atom np . The transformation $nneg(F)$ (“no strong negation”) is defined as $nneg(F) = F' \otimes_m \bigotimes_{p \in \sigma} \neg_s (p \otimes_l np)$.

For any valuation V of signature σ , define the valuation $nneg(V)$ of $\sigma \cup \{np \mid p \in \sigma\}$ as

$$\begin{cases} nneg(V)(w, p) = [V^-(w, p), 1] \\ nneg(V)(w, np) = [1 - V^+(w, p), 1] \end{cases}$$

for all atoms $p \in \sigma$. Clearly, for every valuation V of σ , there exists a corresponding interpretation $I_{nneg(V)}$ of $\sigma \cup \{np \mid p \in \sigma\}$. On the other hand, there exists an interpretation I of $\sigma \cup \{np \mid p \in \sigma\}$ for which there is no corresponding valuation V of σ such that $I = I_{nneg(V)}$.

Example 9. Suppose $\sigma = \{p\}$. For the valuation V such that $V(w, p) = [0.2, 0.7]$, $nneg(V)$ is a valuation of $\{p, np\}$ such that

$$nneg(V)(w, p) = [0.2, 1] \text{ and } nneg(V)(w, np) = [0.3, 1].$$

Further, $I_{nneg(V)}$ is an interpretation of $\{p, np\}$ such that

$$I_{nneg(V)}(p) = 0.2 \text{ and } I_{nneg(V)}(np) = 0.3.$$

On the other hand, the interpretation $I = \{(p, 0.6), (np, 0.8)\}$ of $\sigma \cup \{np \mid p \in \sigma\}$ has no corresponding valuation V of σ such that $I = I_{nneg(V)}$ because $[0.6, 0.2]$ is not a valid valuation.

The following proposition asserts that strong negation can be eliminated in favor of new atoms, extending the well-known results in Boolean stable models [Ferraris *et al.*, 2011, Section 8] to fuzzy formulas.

Proposition 3. For any fuzzy formula F that may contain strong negation, a valuation V is an equilibrium model of F iff $nneg(V)$ is an equilibrium model of $nneg(F)$.

Proposition 3 allows us to extend the 1-1 correspondence between fuzzy equilibrium models and fuzzy stable models in Theorem 6 to any formula that contains strong negation.

Theorem 7. *For any fuzzy formula F of signature σ that may contain strong negation,*

- (a) *A valuation V of σ is a fuzzy equilibrium model of F iff $V(h, p) = V(t, p)$ for all atoms p in σ and $I_{nneg(V)}$ is a stable model of $nneg(F)$ relative to $\sigma \cup \{np \mid p \in \sigma\}$.*
- (b) *An interpretation I of $\sigma \cup \{np \mid p \in \sigma\}$ is a stable model of $nneg(F)$ relative to $\sigma \cup \{np \mid p \in \sigma\}$ iff $I = I_{nneg(V)}$ for some fuzzy equilibrium model V of F .*

Example 10. *For fuzzy formula $F = (0.2 \rightarrow_r p) \otimes_m (0.3 \rightarrow_r \sim p)$, formula $nneg(F)$ is*

$$(0.2 \rightarrow_r p) \otimes_m (0.3 \rightarrow_r np) \otimes_m \neg_s(p \otimes_l np).$$

One can check that the valuation V defined as $V(w, p) = [0.2, 0.7]$ is the only equilibrium model of F , and the interpretation $I_{nneg(V)} = \{(p, 0.2), (np, 0.3)\}$ is the only fuzzy stable model of $nneg(F)$.

This idea of eliminating strong negation in favor of new atoms was used in Examples 3 and 4.

The correspondence between fuzzy equilibrium models and fuzzy stable models indicates that the complexity analysis for fuzzy equilibrium logic applies to fuzzy stable models as well. In [Schockaert *et al.*, 2012] it is shown that deciding whether a formula has a fuzzy equilibrium model is Σ_2^P -hard, which applies to the fuzzy stable model semantics as well. The same problem for a normal FASP programs, which can be identified with a special case of the fuzzy stable model semantics as shown in Section 6.1, is NP-hard. Complexity analyses for other special cases based on restrictions on fuzzy operators, or the presence of cycles in a program have been studied in [Blondeel *et al.*, 2014].

7 Some Properties of Fuzzy Stable Models

In this section, we show that several well-known properties of the Boolean stable model semantics can be naturally extended to the fuzzy stable model semantics.

7.1 Theorem on Constraints

In answer set programming, constraints—rules with \perp in the head—play an important role in view of the fact that adding a constraint eliminates the stable models that “violate” the constraint. The following theorem is the counterpart of Theorem 3 from [Ferraris *et al.*, 2011] for fuzzy propositional formulas.

Theorem 8. *For any fuzzy formulas F and G , I is a stable model of $F \otimes \neg G$ (relative to \mathbf{p}) if and only if I is a stable model of F (relative to \mathbf{p}) and $I \models \neg G$.*

Example 11. *Consider $F = (\neg_s p \rightarrow_r q) \otimes_m (\neg_s q \rightarrow_r p) \otimes_m \neg_s p$. Formula F has only one stable model $I = \{(p, 0), (q, 1)\}$, which is the only stable model of $(\neg_s p \rightarrow_r q) \otimes_m (\neg_s q \rightarrow_r p)$ that satisfies $\neg_s p$.*

7.2 Theorem on Choice Formulas

In the Boolean stable model semantics, formulas of the form $p \vee \neg p$ are called *choice formulas*, and adding them to the program makes atoms p exempt from minimization. Choice formulas have been shown to be useful in constructing an ASP program in the “Generate-and-Test” style. This section shows their counterpart in the fuzzy stable model semantics.

For any finite set of fuzzy atoms $\mathbf{p} = \{p_1, \dots, p_n\}$, the expression \mathbf{p}^{ch} stands for the choice formula

$$(p_1 \oplus_l \neg_s p_1) \otimes \dots \otimes (p_n \oplus_l \neg_s p_n),$$

where \otimes is any fuzzy conjunction.

The following proposition tells us that choice formulas are tautological.

Proposition 4. *For any fuzzy interpretation I and any finite set \mathbf{p} of fuzzy atoms, $I \models \mathbf{p}^{\text{ch}}$.⁹*

Theorem 9 is an extension of Theorem 2 from [Ferraris *et al.*, 2011].

Theorem 9. (a) *For any real number $y \in [0, 1]$, if I is a y -stable model of F relative to $\mathbf{p} \cup \mathbf{q}$, then I is a y -stable model of F relative to \mathbf{p} .*

(b) *I is a 1-stable model of F relative to \mathbf{p} iff I is a 1-stable model of $F \otimes \mathbf{q}^{\text{ch}}$ relative to $\mathbf{p} \cup \mathbf{q}$.*

Theorem 9 (b) does not hold for an arbitrary threshold y (i.e., if “1-” is replaced with “ y -”). For example, consider $F = \neg_s \neg_s q$ and $I = \{(q, 0.5)\}$. Clearly I is a 0.5-model of F , and thus I is a 0.5-stable model of F relative to \emptyset . However, I is not a 0.5-stable model of $F \otimes_m \{q\}^{\text{ch}} = \neg_s \neg_s q \otimes_m (q \oplus_l \neg_s q)$ relative to $\emptyset \cup \{q\}$, as witnessed by $J = \{(q, 0)\}$.

Since the 1-stable models of F relative to \emptyset are the models of F , it follows from Theorem 9 (b) that the 1-stable models of $F \otimes \sigma^{\text{ch}}$ relative to the whole signature σ are exactly the 1-models of F .

⁹This proposition may not hold if \oplus_l in the choice formula is replaced by an arbitrary fuzzy disjunction. For example, consider using \oplus_m instead. Clearly the interpretation $I = \{(p, 0.5)\} \not\models p \oplus_m \neg_s p$.

Corollary 2. *Let F be a fuzzy formula of a finite signature σ . I is a model of F iff I is a stable model of $F \otimes \sigma^{\text{ch}}$ relative to σ .*

Example 12. *Consider the fuzzy formula $F = \neg_s q \rightarrow_r p$ in Example 1, which has only one stable model $\{(p, 1), (q, 0)\}$, although any interpretation $I = \{(p, x), (q, 1 - x)\}$ is a model of F . In accordance with Corollary 2, we check that any I is a stable model of $G = F \otimes_m (p \oplus_l \neg_s p) \otimes_m (q \oplus_l \neg_s q)$. The reduct G^I is equivalent to*

$$((1 - v_I(q)) \rightarrow_r p) \otimes_m (p \oplus_l (1 - v_I(p))) \otimes_m (q \oplus_l (1 - v_I(q))).$$

It is clear that any interpretation J that satisfies G^I should be such that $v_J(p) \geq v_I(p)$ and $v_J(q) \geq v_I(q)$, so there is no witness to dispute the stability of I .

8 Other Related Work

Several approaches to incorporating graded truth degrees into the answer set programming framework have been proposed. In this paper, we have formally compared our approach to [Schockaert *et al.*, 2012] and [Lukasiewicz, 2006]. Most works consider a special form $h \leftarrow B$ where h is an atom and B is a formula [Vojtás, 2001; Damásio and Pereira, 2001b; Medina *et al.*, 2001; Damásio and Pereira, 2001a]. Among them, [Vojtás, 2001; Damásio and Pereira, 2001b; Medina *et al.*, 2001] allow B to be any arbitrary formula corresponding to an increasing function whose arguments are the atoms appearing in the formula. [Damásio and Pereira, 2001a] allows B to correspond to either an increasing function or a decreasing function. [Madrid and Ojeda-Aciego, 2008] considers the normal program syntax, i.e., each rule is of the form $l_0 \leftarrow l_1 \otimes \dots \otimes l_m \otimes \text{not } l_{m+1} \otimes \dots \otimes \text{not } l_n$, where each l_i is an atom or a strongly negated atom. In terms of semantics, most of the previous works rely on the notion of an immediate consequence operator and relate the fixpoint of this operator to the minimal model of a positive program. Similar to the approach [Lukasiewicz, 2006] has adopted, the answer set of a positive program is defined as its minimal model, while the answer sets of a non-positive program are defined as minimal models of reducts. [Van Nieuwenborgh *et al.*, 2007] presented a semantics based on the notion of an unfounded set. Disjunctive fuzzy answer set programs were also studied in [Blondeel *et al.*, 2014].

It is worth noting that some of the related works have discussed the so-called residuated programs [Vojtás, 2001; Damásio and Pereira, 2001b; Medina *et al.*, 2001; Madrid and Ojeda-Aciego, 2008], where each rule $h \leftarrow B$ is assigned a weight θ , and a rule is satisfied by an interpretation I if $I(h \leftarrow B) \geq \theta$. According to [Damásio and Pereira, 2001b], this class of programs is able to capture many other logic programming paradigms, such as possibilistic logic programming, hybrid probabilistic

logic programming, and generalized annotated logic programming. Furthermore, as shown in [Damásio and Pereira, 2001b], a weighted rule $(h \leftarrow B, \theta)$ can be simulated by $h \leftarrow B \otimes \theta$, where (\otimes, \leftarrow) forms an adjoint pair. Notice that a similar method was used in Theorem 1 in relating y -stable models to 1-stable models.

It is well known in the Boolean stable model semantics that strong negation can be represented in terms of new atoms [Ferraris *et al.*, 2011]. Our adaptation in the fuzzy stable model semantics is similar to the method from [Madrid and Ojeda-Aciego, 2008], in which the consistency of an interpretation is guaranteed by imposing the extra restriction $I(\sim p) \leq \sim I(p)$ for all atom p . Strong negation and consistency have also been studied in [Madrid and Ojeda-Aciego, 2011; Madrid and Ojeda-Aciego, 2009].

In addition to fuzzy answer set programming, there are other approaches developed to handle many-valued logic. For example, [Straccia, 2006] proposed a logic programming framework where each literal is annotated by a real-valued interval, in this way the nonmonotonicity of answer set programming and many-valued logic are combined. Another example is possibilistic logic [Dubois and Prade, 2004], where each propositional symbol is associated with two real values in the interval $[0, 1]$ called necessity degree and possibility degree. Although these semantics handle fuzziness based on quite different ideas, it has been shown that these paradigms can be captured by fuzzy answer set programming [Damásio and Pereira, 2001b].

While the development of FASP solvers has not yet reached the maturity level of ASP solvers, there is an increasing interest recently. [Alviano and Penalzoza, 2013] presented a FASP solver based on answer set approximation operators and a translation to bilevel linear programming presented in [Blondeel *et al.*, 2014]. The implementation in [Mushthofa *et al.*, 2014] is based on a reduction of FASP to ASP. Independent from the work presented here, [Alviano and Penalzoza, 2015] presented another promising FASP solver, named FASP2SMT, that uses SMT solvers based on a translation from FASP into satisfiability modulo theories. A large fragment of the language proposed in this paper can be computed by this solver. The input language allows $\neg_s, \otimes_l, \oplus_l, \otimes_m, \oplus_m$ as fuzzy operators, and rules of the form

$$Head \leftarrow Body$$

where *Head* is $p_1 \odot \dots \odot p_n$ where p_i are atoms or numeric constants, and $\odot \in \{\otimes_l, \oplus_l, \otimes_m, \oplus_m\}$, and *Body* is a nested formula formed from atoms and numeric constants using $\neg_s, \otimes_l, \oplus_l, \otimes_m, \oplus_m$.

Example 4 can be computed by FASP2SMT. Assume a conflict of degree 0.1 between Alice and Bob occurred at step 0, no conflict occurred at step 1, and a conflict of degree 0.5 between Alice and Bob occurred at step 2. Since the current version

of `fast2smt`¹⁰ does not yet support product conjunction \otimes_p , we use Łukasiewicz conjunction \otimes_l in formula F_4 . In the input language of FASP2SMT, “:-” denote \rightarrow_r , “,” denotes \otimes_m , “*” denote \otimes_l , “+” denotes \oplus_l , and “not” denotes \neg_s . Numeric constants begin with “#” symbol, and variables are capitalized. The encoding in the input language of FASP2SMT is shown in Figure 2.

The command line to compute this program is simply:

```
python fasp2smt.py trust.fasp
```

Part of the output from FASP2SMT is shown below:

<code>trust(bob,carol,2)</code>	0.700000	(7.0/10.0)
<code>trust(alice,carol,3)</code>	0.500000	(1.0/2.0)
<code>trust(bob,carol,0)</code>	0.700000	(7.0/10.0)
<code>trust(alice,carol,1)</code>	0.500000	(1.0/2.0)
<code>trust(alice,bob,1)</code>	0.700000	(7.0/10.0)
<code>trust(alice,carol,0)</code>	0.500000	(1.0/2.0)
<code>conflict(alice,bob,2)</code>	0.500000	(1.0/2.0)
<code>trust(alice,bob,0)</code>	0.800000	(4.0/5.0)
<code>trust(alice,carol,2)</code>	0.500000	(1.0/2.0)
<code>trust(alice,bob,2)</code>	0.700000	(7.0/10.0)
<code>conflict(alice,bob,1)</code>	0.0	(0.0)
<code>trust(alice,bob,3)</code>	0.200000	(1.0/5.0)
<code>conflict(alice,bob,0)</code>	0.100000	(1.0/10.0)
<code>trust(bob,carol,3)</code>	0.700000	(7.0/10.0)

The number following each atom gives the truth value of the atom in a decimal notation, and the numbers in the parenthesis give the truth values in fractional forms.

9 Conclusion

We introduced a stable model semantics for fuzzy propositional formulas, which generalizes both the Boolean stable model semantics and fuzzy propositional logic. The syntax is the same as the syntax of fuzzy propositional logic, but the semantics allows us to distinguish *stable models* from non-stable models. The formalism allows highly configurable default reasoning involving fuzzy truth values. The proposed semantics, when we restrict threshold to be 1 and assume all atoms to be subject to minimization, is essentially equivalent to fuzzy equilibrium logic, but is much simpler. To the best of our knowledge, our representation of the commonsense law of inertia involving fuzzy values is new. The representation uses nested fuzzy

¹⁰Downloaded in March 2016


```

% domain
person(alice).      person(bob).      person(carol).
step(0).           step(1).           step(2).           step(3).
next(0,1).         next(1,2).         next(2,3).

% F1: trust is reflexive
trust(X,X,T) :- person(X), step(T).

% F2, F3: UEC
:- trust(X,Y,T) * distrust(X,Y,T).
:- not (trust(X,Y,T) + distrust(X,Y,T)), person(X), person(Y), step(T).

% F4, F5: transitivity of trust
% F4 modified: product t-norm is replaced by Lukasiewicz t-norm since
% the solver does not support it.
trust(X,Z,0) :- trust(X,Y,0) * trust(Y,Z,0),
                person(X), person(Y), person(Z).
distrust(X,Y,0) :- not trust(X,Y,0), person(X), person(Y).

% F6, F7: inertia
trust(X,Y,T2) :- trust(X,Y,T1), not not trust(X,Y,T2), next(T1,T2).
distrust(X,Y,T2) :- distrust(X,Y,T1), not not distrust(X,Y,T2), next(T1,T2).

% F8: effect of conflict
distrust(X,Y,T2) :- conflict(X,Y,T1) + distrust(X,Y,T1), next(T1,T2).

% initial State
trust(alice,bob,0) :- #0.8.
trust(bob,carol,0) :- #0.7.

% action
conflict(alice,bob,0) :- #0.1.
conflict(alice,bob,2) :- #0.5.

```

Figure 2: Trust Example in the Input Language of FASP2SMT

operators, which are not available in other fuzzy ASP semantics with restricted syntax.

We showed that several traditional results in answer set programming can be naturally extended to this formalism, and expect that more results can be carried over. Also, it would be possible to generalize the semantics to the first-order level, similar to the way the Boolean stable model semantics was generalized in [Ferraris *et al.*, 2011].

References

- [Alviano and Penaloza, 2013] Mario Alviano and Rafael Penaloza. Fuzzy answer sets approximations. *Theory and Practice of Logic Programming*, 13(4-5):753–767, 2013.
- [Alviano and Penaloza, 2015] Mario Alviano and Rafael Penaloza. Fuzzy answer set computation via satisfiability modulo theories. *Theory and Practice of Logic Programming*, 15(4-5):588–603, 2015.
- [Blondeel *et al.*, 2014] Marjon Blondeel, Steven Schockaert, Dirk Vermeir, and Martine De Cock. Complexity of fuzzy answer set programming under Łukasiewicz semantics. *International Journal of Approximate Reasoning*, 55(9):1971–2003, 2014.
- [Damásio and Pereira, 2001a] Carlos Viegas Damásio and Luís Moniz Pereira. Antitonic logic programs. In *Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*, pages 379–392, 2001.
- [Damásio and Pereira, 2001b] Carlos Viegas Damásio and Luís Moniz Pereira. Monotonic and residuated logic programs. In Salem Benferhat and Philippe Besnard, editors, *EC-SQARU*, volume 2143 of *Lecture Notes in Computer Science*, pages 748–759. Springer, 2001.
- [Dubois and Prade, 2004] Didier Dubois and Henri Prade. Possibilistic logic: a retrospective and prospective view. *Fuzzy Sets and Systems*, 144(1):3–23, 2004.
- [Ferraris *et al.*, 2006] Paolo Ferraris, Joohyung Lee, and Vladimir Lifschitz. A generalization of the Lin-Zhao theorem. *Annals of Mathematics and Artificial Intelligence*, 47:79–101, 2006.
- [Ferraris *et al.*, 2009] Paolo Ferraris, Joohyung Lee, Vladimir Lifschitz, and Ravi Palla. Symmetric splitting in the general theory of stable models. In *Proceedings of International Joint Conference on Artificial Intelligence (IJCAI)*, pages 797–803. AAAI Press, 2009.
- [Ferraris *et al.*, 2011] Paolo Ferraris, Joohyung Lee, and Vladimir Lifschitz. Stable models and circumscription. *Artificial Intelligence*, 175:236–263, 2011.
- [Ferraris, 2005] Paolo Ferraris. Answer sets for propositional theories. In *Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*, pages 119–131, 2005.
- [Hajek, 1998] Peter Hajek. *Mathematics of Fuzzy Logic*. Kluwer, 1998.

- [Janssen *et al.*, 2012] Jeroen Janssen, Dirk Vermeir, Steven Schockaert, and Martine De Cock. Reducing fuzzy answer set programming to model finding in fuzzy logics. *Theory and Practice of Logic Programming*, 12(06):811–842, 2012.
- [Lee and Wang, 2014a] Joohyung Lee and Yi Wang. Stable models of fuzzy propositional formulas. In *Working Notes of the 1st Workshop on Logics for Reasoning about Preferences, Uncertainty, and Vagueness (PRUV)*, 2014.
- [Lee and Wang, 2014b] Joohyung Lee and Yi Wang. Stable models of fuzzy propositional formulas. In *Proceedings of European Conference on Logics in Artificial Intelligence (JELIA)*, pages 326–339, 2014.
- [Lee *et al.*, 2008] Joohyung Lee, Vladimir Lifschitz, and Ravi Palla. A reductive semantics for counting and choice in answer set programming. In *Proceedings of the AAAI Conference on Artificial Intelligence (AAAI)*, pages 472–479, 2008.
- [Lifschitz and Turner, 1999] Vladimir Lifschitz and Hudson Turner. Representing transition systems by logic programs. In *Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*, pages 92–106, 1999.
- [Lifschitz, 2002] Vladimir Lifschitz. Answer set programming and plan generation. *Artificial Intelligence*, 138:39–54, 2002.
- [Lifschitz, 2008] Vladimir Lifschitz. What is answer set programming? In *Proceedings of the AAAI Conference on Artificial Intelligence*, pages 1594–1597. MIT Press, 2008.
- [Lukasiewicz, 2006] Thomas Lukasiewicz. Fuzzy description logic programs under the answer set semantics for the semantic web. In Thomas Eiter, Enrico Franconi, Ralph Hodgson, and Susie Stephens, editors, *RuleML*, pages 89–96. IEEE Computer Society, 2006.
- [Madrid and Ojeda-Aciego, 2008] Nicolás Madrid and Manuel Ojeda-Aciego. Towards a fuzzy answer set semantics for residuated logic programs. In *Web Intelligence/IAT Workshops*, pages 260–264. IEEE, 2008.
- [Madrid and Ojeda-Aciego, 2009] Nicolás Madrid and Manuel Ojeda-Aciego. On coherence and consistence in fuzzy answer set semantics for residuated logic programs. In Vito Di Gesù, Sankar K. Pal, and Alfredo Petrosino, editors, *WILF*, volume 5571 of *Lecture Notes in Computer Science*, pages 60–67. Springer, 2009.
- [Madrid and Ojeda-Aciego, 2011] Nicolás Madrid and Manuel Ojeda-Aciego. Measuring inconsistency in fuzzy answer set semantics. *IEEE T. Fuzzy Systems*, 19(4):605–622, 2011.
- [Medina *et al.*, 2001] Jesús Medina, Manuel Ojeda-Aciego, and Peter Vojtás. Multi-adjoint logic programming with continuous semantics. In *Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR)*, pages 351–364, 2001.
- [Mushthofa *et al.*, 2014] Mushthofa Mushthofa, Steven Schockaert, and Martine De Cock. A finite-valued solver for disjunctive fuzzy answer set programs. In *21st European conference on Artificial Intelligence (ECAI 2014)*, volume 263, pages 645–650. IOS Press, 2014.
- [Schockaert *et al.*, 2012] Steven Schockaert, Jeroen Janssen, and Dirk Vermeir. Fuzzy equilibrium logic: Declarative problem solving in continuous domains. *ACM Transactions on Computational Logic (TOCL)*, 13(4):33, 2012.
- [Straccia, 2006] Umberto Straccia. Annotated answer set programming. In *Proceedings of*

the 11th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, pages 1212–1219, 2006.

[Van Nieuwenborgh *et al.*, 2007] Davy Van Nieuwenborgh, Martine De Cock, and Dirk Vermeir. An introduction to fuzzy answer set programming. *Annals of Mathematics and Artificial Intelligence*, 50(3-4):363–388, 2007.

[Vojtás, 2001] Peter Vojtás. Fuzzy logic programming. *Fuzzy Sets and Systems*, 124(3):361–370, 2001.

A Proofs

A.1 Proof of Proposition 1

Lemma 2. *For any fuzzy conjunction \otimes , we have $\otimes(x, y) \leq x$ and $\otimes(x, y) \leq y$.*

Proof. By the conditions imposed on fuzzy conjunctions, we have $\otimes(x, y) \leq \otimes(x, 1) = x$ and $\otimes(x, y) \leq \otimes(1, y) = y$. ■

Proposition 1. *For any interpretations I and J such that $J \leq^{\mathbf{P}} I$, it holds that*

$$v_J(F^I) \leq v_I(F).$$

Proof. By induction on F .

- F is an atom p . $v_J(F^I) = J(p)$ and $v_I(F) = I(p)$. Clear from the assumption $J \leq^{\mathbf{P}} I$.
- F is a numeric constant c . Clearly, $v_J(F^I) = c = v_I(F)$.
- F is $\neg G$. $v_J(F^I) = v_J(v_I(\neg G)) = v_I(\neg G) = v_I(F)$.
- F is $G \odot H$, where \odot is \otimes or \oplus . $v_J(F^I) = v_J(G^I) \odot v_J(H^I)$. By I.H. we have $v_J(G^I) \leq v_I(G)$ and $v_J(H^I) \leq v_I(H)$. Since \otimes and \oplus are both increasing, we have $v_J(F^I) = v_J(G^I) \odot v_J(H^I) \leq v_I(G) \odot v_I(H) = v_I(F)$.
- F is $G \rightarrow H$. $v_J(F^I) = (v_J(G^I) \rightarrow v_J(H^I)) \otimes_m v_I(F)$. By Lemma 2, $v_J(F^I) \leq v_I(F)$ ■

A.2 Proof of Proposition 2

Lemma 3. *For any (fuzzy) formula F and (fuzzy) interpretation I , we have $v_I(F) = v_I(F^I)$.*

Proof. By induction on F .

- F is an atom p or a numeric constant. Clear from the fact $F^I = F$.
- F is $\neg G$. Then we have $v_I(F) = v_I(\neg G) = v_I(v_I(\neg G)) = v_I(F^I)$.
- F is $G \odot H$, where \odot is \otimes , \oplus , or \rightarrow . Then $F^I = (G^I \odot H^I) \otimes_m v_I(F \odot G)$. By I.H., we have $v_I(G) = v_I(G^I)$ and $v_I(H) = v_I(H^I)$. So we have

$$\begin{aligned}
 v_I(F^I) &= \min \left\{ v_I(G^I \odot H^I), v_I(G \odot H) \right\} \\
 &= \min \left\{ \odot(v_I(G^I), v_I(H^I)), \odot(v_I(G), v_I(H)) \right\} \\
 &= \min \left\{ \odot(v_I(G), v_I(H)), \odot(v_I(G), v_I(H)) \right\} \\
 &= \odot(v_I(G), v_I(H)) \\
 &= v_I(F).
 \end{aligned}$$

■

Proposition 2 is an immediate corollary to Lemma 3.

Proposition 2. *A (fuzzy) interpretation I satisfies a (fuzzy) formula F if and only if I satisfies F^I .*

A.3 Proof of Theorem 1

Lemma 4. *For any fuzzy formula F , any interpretation I , and any implication \rightarrow that satisfies $\rightarrow(x, y) = 1$ iff $y \geq x$, we have that I is a y -model of F iff I is a 1-model of $y \rightarrow F$.*

Proof. By definition, $I \models_y F$ means that $v_I(F) \geq y$. Since $\rightarrow(x, y) = 1$ iff $y \geq x$, $v_I(F) \geq y$ iff $\rightarrow(y, v_I(F)) = 1$ iff $v_I(y \rightarrow F) = 1$ iff $I \models_1 (y \rightarrow F)$. ■

Theorem 1. *For any fuzzy formula F , an interpretation I is a y -stable model of F relative to \mathbf{p} iff I is a 1-stable model of $y \rightarrow F$ relative to \mathbf{p} as long as the implication \rightarrow satisfies the condition $\rightarrow(x, y) = 1$ iff $y \geq x$.*

Proof.

I is a y -stable model of F relative to \mathbf{p}

iff

$$I \models_y F \text{ and there is no } J <^{\mathbf{P}} I \text{ such that } J \models_y F^I$$

iff (by Lemma 4)

$$I \models_1 y \rightarrow F \text{ and there is no } J <^{\mathbf{P}} I \text{ such that } J \models_1 (y \rightarrow F^I)$$

iff (since $v_I(y \rightarrow F) = 1$)

$$I \models_1 y \rightarrow F \text{ and there is no } J <^{\mathbf{P}} I \text{ such that } J \models_1 (y \rightarrow F^I) \otimes_m v_I(y \rightarrow F)$$

iff (since $(y \rightarrow F^I) \otimes_m v_I(y \rightarrow F) = (y \rightarrow F)^I$)

$$I \text{ is a 1-stable model of } y \rightarrow F \text{ relative to } \mathbf{p}.$$

■

A.4 Proof of Theorem 2

Lemma 5. *For any formula F and any interpretations I and J such that $J \leq^{\mathbf{P}} I$, $v_{I \cup J_{\mathbf{q}}} (F^*(\mathbf{q})) = v_J(F^I)$.*

Proof. By induction on F .

- F is a numeric constant c , or an atom not in \mathbf{p} . Immediate from the fact that J and I agree on $F^*(\mathbf{q}) = F = F^I$.
- F is an atom $p_i \in \mathbf{p}$. $F^*(\mathbf{q}) = q_i$ and $F^I = p_i$. Clear from the fact that $v_{I \cup J_{\mathbf{q}}}(q_i) = v_J(p_i)$.
- F is $\neg G$. $v_{I \cup J_{\mathbf{q}}}((\neg G)^*(\mathbf{q})) = v_I(\neg G) = v_J(v_I(\neg G)) = v_J(F^I)$.
- F is $G \odot H$, where \odot is \otimes or \oplus . Immediate by I.H. on G and H .
- F is $G \rightarrow H$. By I.H., $v_{I \cup J_{\mathbf{q}}}(G^*(\mathbf{q})) = v_J(G^I)$ and $v_{I \cup J_{\mathbf{q}}}(H^*(\mathbf{q})) = v_J(H^I)$. $F^*(\mathbf{q})$ is $(G^*(\mathbf{q}) \rightarrow H^*(\mathbf{q})) \otimes_m (G \rightarrow H)$, and F^I is $(G^I \rightarrow H^I) \otimes_m v_I(G \rightarrow H)$. Then the claim is immediate from I.H. ■

■

Theorem 2. *A fuzzy interpretation I is a fuzzy stable model of F relative to \mathbf{p} iff*

- $I \models F$, and

- there is no fuzzy interpretation J such that $J <^{\mathbf{P}} I$ and $I \cup J_{\mathbf{q}}^{\mathbf{P}} \models F^*(\mathbf{q})$.

Proof.

I is a fuzzy stable model of F relative to \mathbf{p}

iff

$I \models F$ and there is no $J <^{\mathbf{P}} I$ such that $J \models F^I$

iff (by Lemma 5)

$I \models F$ and there is no $J <^{\mathbf{P}} I$ such that $I \cup J_{\mathbf{q}}^{\mathbf{P}} \models F^*(\mathbf{q})$.

■

A.5 Proofs of Theorems 3 and 4

The following lemma immediately follows from Lemma 2.

Lemma 6. For any fuzzy conjunction \otimes , $\otimes(x, y) = 1$ if and only if $x = 1$ and $y = 1$.

Define the mapping $defuz(I)$ that maps a fuzzy interpretation $I = \{(p_1, x_1), \dots, (p_n, x_n)\}$ to a classical interpretation such that $defuz(I) = \{p_i \mid (p_i, 1) \in I\}$.

Lemma 7. for any fuzzy interpretation I and any classical propositional formula F ,

- (i) if $v_I(F^{fuzzy}) = 1$, then $defuz(I) \models F$, and
- (ii) if $v_I(F^{fuzzy}) = 0$, then $defuz(I) \not\models F$.

Proof. We prove by induction on F .

- F is an atom p . (i) Suppose $I \models_1 F^{fuzzy}$. Then $v_I(p) = 1$, and thus $p \in defuz(I)$. So $defuz(I) \models F$. (ii) Suppose $v_I(F^{fuzzy}) = 0$. Then $v_I(p) = 0$, and thus $p \notin defuz(I)$. So $defuz(I) \not\models F$.
- F is \perp . (i) There is no interpretation that satisfies $F^{fuzzy} = 0$ to the degree 1. So the claim is trivially true. (ii) Since no interpretation satisfies F , $defuz(I) \not\models F$.
- F is \top . (i) All interpretations satisfy F . So $defuz(I) \models F$. (ii) There is no interpretation I such that $v_I(F^{fuzzy}) = 0$. So (ii) is trivially true.

- F is $\neg G$. Then F^{fuzzy} is $\neg_s G^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$. Then $v_I(F^{fuzzy}) = 1 - v_I(G^{fuzzy}) = 1$, so $v_I(G^{fuzzy}) = 0$. By I.H., $defuz(I) \not\models G$, and thus $defuz(I) \models F$. (ii) Suppose $v_I(F^{fuzzy}) = 0$, Then $1 - v_I(G^{fuzzy}) = 0$, $v_I(G^{fuzzy}) = 1$, i.e., $I \models_1 v_I(G^{fuzzy})$. By I.H., $defuz(I) \models G$, and thus $defuz(I) \not\models F$.
- F is $G \wedge H$. Then F^{fuzzy} is $G^{fuzzy} \otimes_m H^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$. By Lemma 6, $v_I(G^{fuzzy}) = v_I(H^{fuzzy}) = 1$, i.e., $I \models_1 G^{fuzzy}$ and $I \models_1 H^{fuzzy}$. By I.H., $defuz(I) \models G$ and $defuz(I) \models H$. It follows that $defuz(I) \models G \wedge H = F$. (ii) Suppose $v_I(F^{fuzzy}) = \min(v_I(G^{fuzzy}), v_I(H^{fuzzy})) = 0$. Then $v_I(G^{fuzzy}) = 0$ or $v_I(H^{fuzzy}) = 0$. By I.H., $defuz(I) \not\models G$ or $defuz(I) \not\models H$. It follows that $defuz(I) \not\models G \wedge H = F$.
- F is $G \vee H$. Then F^{fuzzy} is $G^{fuzzy} \oplus_m H^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$, and as the disjunction is defined as $\oplus_m(x, y) = \max(x, y)$, $v_I(G^{fuzzy}) = 1$ or $v_I(H^{fuzzy}) = 1$, i.e., $I \models_1 G^{fuzzy}$ or $I \models_1 H^{fuzzy}$. By I.H., $defuz(I) \models G$ or $defuz(I) \models H$. It follows that $defuz(I) \models G \vee H = F$. (ii) Suppose $v_I(F^{fuzzy}) = \max(v_I(G^{fuzzy}), v_I(H^{fuzzy})) = 0$. Then $v_I(G^{fuzzy}) = 0$ and $v_I(H^{fuzzy}) = 0$. By I.H., $defuz(I) \not\models G$ and $defuz(I) \not\models H$. It follows that $defuz(I) \not\models G \vee H = F$.
- F is $G \rightarrow H$. Then F^{fuzzy} is $G^{fuzzy} \rightarrow_s H^{fuzzy}$. (i) Suppose $I \models_1 F^{fuzzy}$. $v_I(F^{fuzzy}) = \max(1 - v_I(G^{fuzzy}), v_I(H^{fuzzy})) = 1$, and thus $v_I(G^{fuzzy}) = 0$ or $v_I(H^{fuzzy}) = 1$.¹¹ By I.H., $defuz(I) \not\models G$ or $defuz(I) \models H$. It follows that $defuz(I) \models G \rightarrow H = F$. (ii) Suppose $v_I(F^{fuzzy}) = \max\{1 - v_I(G^{fuzzy}), v_I(H^{fuzzy})\} = 0$. Then $v_I(G^{fuzzy}) = 1$ and $v_I(H^{fuzzy}) = 0$. By I.H., $defuz(I) \models G$ and $defuz(I) \not\models H$. Therefore $defuz(I) \not\models G \rightarrow H$.

■

Lemma 8. *For any classical interpretation I and any classical propositional formula F , $I \models F$ if and only if $I^{fuzzy} \models F^{fuzzy}$.*

Proof. By induction on F . ■

¹¹This does not hold for an arbitrary choice of implication. For example, consider $\rightarrow_l(x, y) = \min(1 - x + y, 1)$, then from $I \models_1 G \rightarrow_l H$, we can only conclude $v_I(H) \geq v_I(G)$. Furthermore, under this choice of implication, the modified statement of the lemma does not hold. A counterexample is $F = \neg_s p \rightarrow_l q$, $I = \{(p, 0.5), (q, 0.6)\}$. Clearly $I \models_1 F^{fuzzy}$ but $defuz(I) = \emptyset \not\models F$.

Theorem 3. *For any classical propositional formula F and any classical propositional interpretation I , I is a stable model of F relative to \mathbf{p} iff I^{fuzzy} is a stable model of F^{fuzzy} relative to \mathbf{p} .*

Proof. (\Rightarrow) Suppose I is a stable model of F relative to \mathbf{p} . From the fact that $I \models F$, by Lemma 8, $I^{fuzzy} \models F^{fuzzy}$.

Next we show that there is no fuzzy interpretation $J <^{\mathbf{p}} I^{fuzzy}$ such that $J \models (F^{fuzzy})^I$. Suppose, for the sake of contradiction, that there exists such J . Since $J \models (F^{fuzzy})^I$, or equivalently, $J \models (F^I)^{fuzzy}$, by Lemma 7, we get $defuz(J) \models F^I$.

Since $J <^{\mathbf{p}} I^{fuzzy}$, J and I^{fuzzy} agree on all atoms not in \mathbf{p} . Since I^{fuzzy} assigns either 0 or 1 to each atom, it follows that J , as well as $defuz(J)$, assigns the same truth values as I^{fuzzy} to atoms not in \mathbf{p} . The construction of $defuz(J)$ guarantees that $defuz(J)^{fuzzy} \leq^{\mathbf{p}} J <^{\mathbf{p}} I^{fuzzy}$. Since both $defuz(J)^{fuzzy}$ and I^{fuzzy} assigns either 0 or 1 to each atom, there is at least one atom $p \in \mathbf{p}$ such that $v_{defuz(J)^{fuzzy}}(p) = 0$ and $v_{I^{fuzzy}}(p) = 1$, and consequently $defuz(J)(p) = \text{FALSE}$ and $I(p) = \text{TRUE}$. So $defuz(J) <^{\mathbf{p}} I$, and together with the fact that $defuz(J) \models F^I$, it follows that I is not a stable model of F , which is a contradiction. Thus, there is no such J , from which we conclude that I^{fuzzy} is a stable model of F^{fuzzy} relative to \mathbf{p} .

(\Leftarrow) Suppose I^{fuzzy} is a stable model of a fuzzy formula F^{fuzzy} relative to \mathbf{p} . Then $I^{fuzzy} \models F^{fuzzy}$. By Lemma 8, $I \models F$.

Next we show there is no $J <^{\mathbf{p}} I$ such that $J \models F^I$. Suppose, for the sake of contradiction, that there exists such J . Then by Lemma 8, $J^{fuzzy} \models (F^I)^{fuzzy}$, and obviously $J^{fuzzy} <^{\mathbf{p}} I^{fuzzy}$. It follows that I^{fuzzy} is not a stable model of F^{fuzzy} relative to \mathbf{p} , which is a contradiction. So there is no such J , from which we conclude that I is a stable model of F relative to \mathbf{p} . ■

The proof of Theorem 4 is the same as the right-to-left direction of the proof of Theorem 3. Notice that the left-to-right direction of the proof of Theorem 3 relies on Lemma 7, which assumes the particular selection of fuzzy operators, so this direction does not apply to the setting of Theorem 4.

A.6 Proof of Theorem 5

Theorem 5. *For any normal FASP program $\Pi = \{r_1, \dots, r_n\}$, let F be the fuzzy formula $r_1 \otimes \dots \otimes r_n$, where \otimes is any fuzzy conjunction. An interpretation I is a fuzzy answer set of Π in the sense of [Lukasiewicz, 2006] if and only if I is a stable model of F .*

Proof. By Π^I we denote the reduct of Π relative to I as defined in [Lukasiewicz, 2006], which is also reviewed in Section 6.1.

(\Rightarrow) Suppose I is a fuzzy answer set of Π . By definition, $I \models \Pi$, and thus $v_I(r_i) = 1$ for all $r_i \in \Pi$. So

$$v_I(F) = v_I(r_1 \otimes \dots \otimes r_n) = 1,$$

i.e., $I \models F$.

Next we show that there is no $J <^\sigma I$ such that $J \models F^I$, where σ is the underlying signature. For each $r_i = a \leftarrow_r b_1 \otimes \dots \otimes b_m \otimes \neg b_{m+1} \otimes \dots \otimes \neg b_n$,

$$r_i^I = v_I(r_i) \otimes_m (a \leftarrow_r b_1 \otimes \dots \otimes b_m \otimes v_I(\neg_s b_{m+1}) \otimes \dots \otimes v_I(\neg_s b_n)).$$

Suppose, for the sake of contradiction, that there exists an interpretation $J <^\sigma I$ such that $J \models F^I$. Then, for all $r_i \in \Pi$, $J \models r_i^I$, i.e.,

$$J \models v_I(r_i) \otimes_m (a \leftarrow_r b_1 \otimes \dots \otimes b_m \otimes v_I(\neg_s b_{m+1}) \otimes \dots \otimes v_I(\neg_s b_n)).$$

It follows that

$$J \models a \leftarrow_r b_1 \otimes \dots \otimes b_m \otimes v_I(\neg_s b_{m+1}) \otimes \dots \otimes v_I(\neg_s b_n).$$

So $J \models \Pi^I$. Together with the fact $J <^\sigma I$, this contradicts that I is a fuzzy answer set of Π . So there is no such J , from which we conclude that I is a stable model of F .

(\Leftarrow) Suppose I is a stable model of F . From the fact that $I \models F$, it follows that $v_I(r_i) = 1$ for all $r_i \in \Pi$. Thus $I \models \Pi$.

Next we show that there is no $J <^\sigma I$ such that $J \models \Pi^I$, where σ is the underlying signature. The reduct Π^I contains the following rule for each original rule $r_i \in \Pi$:

$$a \leftarrow b_1 \otimes \dots \otimes b_m \otimes v_I(\neg b_{m+1}) \otimes \dots \otimes v_I(\neg b_n).$$

Suppose, for the sake of contradiction, that there exists $J <^\sigma I$ such that $J \models \Pi^I$. Then for each rule $r_i \in \Pi$,

$$J \models a \leftarrow_r b_1 \otimes \dots \otimes b_m \otimes v_I(\neg_s b_{m+1}) \otimes \dots \otimes v_I(\neg_s b_n).$$

As $I \models \Pi$, for all $r_i \in \Pi$, $v_I(r_i) = 1$, so

$$J \models v_I(r_i) \otimes_m (a \leftarrow_r b_1 \otimes \dots \otimes b_m \otimes \neg_s v_I(b_{m+1}) \otimes \dots \otimes v_I(\neg_s b_n))$$

or equivalently, $J \models r_i^I$ for each $r_i \in \Pi$. Therefore, $J \models (r_1 \otimes \dots \otimes r_n)^I$, i.e., $J \models F^I$. Together with the fact $J <^\sigma I$, this contradicts that I is a stable model of F . So there is no such J , from which we conclude that I is a fuzzy answer set of Π . ■

A.7 Proof of Theorem 6

Let σ be a signature, and let F be a fuzzy formula of signature σ that does not contain strong negation.

For any two interpretations I and J such that $J \leq^\sigma I$, we define the fuzzy N5 valuation $\mathbb{V}_{J,I}$ as follows. For every atom $p \in \sigma$,

- $\mathbb{V}_{J,I}(h, p) = [v_J(p), 1]$, and
- $\mathbb{V}_{J,I}(t, p) = [v_I(p), 1]$.

It can be seen that $\mathbb{V}_{J,I}$ is always a valid valuation as long as $J \leq^\sigma I$. Clearly, $\mathbb{V}_{I,I} = I$ for any interpretation I .

Lemma 9. *For any interpretations I and J such that $J \leq^\sigma I$, it holds that $v_I(F) = \mathbb{V}_{J,I}^-(t, F)$.*

Proof. By induction on F .

- F is an atom p . Then $v_I(F) = v_I(p) = \mathbb{V}_{J,I}^-(t, p) = \mathbb{V}_{J,I}^-(t, F)$.
- F is a numeric constant c . Then $v_I(F) = c = \mathbb{V}_{J,I}^-(t, c) = \mathbb{V}_{J,I}^-(t, F)$.
- F is $\neg G$. Then $v_I(F) = \neg(v_I(G))$. By I.H., $v_I(F) = \neg(\mathbb{V}_{J,I}^-(t, G)) = \mathbb{V}_{J,I}^-(t, F)$.
- F is $G \odot H$, where \odot is \otimes , \oplus or \rightarrow . Then $v_I(F) = v_I(G) \odot v_I(H)$. By I.H., $v_I(F) = \mathbb{V}_{J,I}^-(t, G) \odot \mathbb{V}_{J,I}^-(t, H) = \mathbb{V}_{J,I}^-(t, F)$.

■

The following is a corollary to Lemma 9.

Corollary 3. *$I \models F$ if and only if $\mathbb{V}_{I,I}$ is a model of F .*

Proof. By Lemma 9, $v_I(F) = \mathbb{V}_{I,I}^-(t, F)$. Furthermore, $\mathbb{V}_{I,I}^-(h, F) = \mathbb{V}_{I,I}^-(t, F)$ can be proven by induction. ■

Lemma 10. *For any interpretations I and J such that $J \leq^\sigma I$, it holds that $v_J(F^I) = \mathbb{V}_{J,I}^-(h, F)$.*

Proof. By induction on F .

- F is an atom p or a numeric constant. Then $v_J(F^I) = v_J(F) = \mathbb{V}_{J,I}^-(h, F)$.

- F is $\neg G$. Then $v_J(F^I) = \neg v_I(G)$, and by Lemma 9, $\neg v_I(G) = \neg \mathbb{V}_{J,I}^-(t, G) = \mathbb{V}_{J,I}^-(h, F)$.
- F is $G \odot H$, where \odot is \otimes or \oplus . Then

$$\begin{aligned}
 v_J(F^I) &= v_J(G^I \odot H^I) = v_J(G^I) \odot v_J(H^I) \\
 &= \text{(by I.H.)} \\
 &\quad \mathbb{V}_{J,I}^-(h, G) \odot \mathbb{V}_{J,I}^-(h, H) \\
 &= \mathbb{V}_{J,I}^-(h, G \odot H) = \mathbb{V}_{J,I}^-(h, F).
 \end{aligned}$$

- F is $G \rightarrow H$. Then

$$\begin{aligned}
 v_J(F^I) &= v_J((G^I \rightarrow H^I) \otimes_m v_I(G \rightarrow H)) \\
 &= \min(v_J(G^I \rightarrow H^I), v_I(G \rightarrow H)) \\
 &= \text{(by I.H. and Lemma 9)} \\
 &\quad \min(\mathbb{V}_{J,I}^-(h, G) \rightarrow \mathbb{V}_{J,I}^-(h, H), \mathbb{V}_{J,I}^-(t, G) \rightarrow \mathbb{V}_{J,I}^-(t, H)) \\
 &= \mathbb{V}_{J,I}^-(h, G \rightarrow H) = \mathbb{V}_{J,I}^-(h, F).
 \end{aligned}$$

■

The following is a corollary to Lemma 10.

Corollary 4. *For any interpretations I and J such that $J \leq^\sigma I$, $J \models F^I$ if and only if $\mathbb{V}_{J,I}$ is a model of F .*

Lemma 11. *For two interpretations I and J , it holds that $\mathbb{V}_{J,I} \prec \mathbb{V}_{I,I}$ if and only if $J < I$.*

Proof. (\Rightarrow) Suppose $\mathbb{V}_{J,I} \prec \mathbb{V}_{I,I}$. For every atom a , $\mathbb{V}_{I,I}(h, a) \subseteq \mathbb{V}_{J,I}(h, a)$, which means $\mathbb{V}_{J,I}^-(h, a) \leq \mathbb{V}_{I,I}^-(h, a)$. So $J \leq I$. Furthermore, there is at least one atom p satisfying $\mathbb{V}_{I,I}(h, p) \subset \mathbb{V}_{J,I}(h, p)$. By the definition of $\mathbb{V}_{I,I}$ and $\mathbb{V}_{J,I}$, $\mathbb{V}_{I,I}^+(h, a) = \mathbb{V}_{J,I}^+(h, a) = 1$ for all a . Consequently, $\mathbb{V}_{J,I}^-(h, p) < \mathbb{V}_{I,I}^-(h, p)$, which means $v_J(p) < v_I(p)$. So $J < I$.

(\Leftarrow) Suppose $J < I$. Then for every atom a , $v_J(a) \leq v_I(a)$. It follows that $\mathbb{V}_{J,I}^-(h, a) \leq \mathbb{V}_{I,I}^-(h, a)$ for all a , and as $\mathbb{V}_{I,I}^+(h, a) = \mathbb{V}_{J,I}^+(h, a) = 1$ for all a , we conclude $\mathbb{V}_{I,I}(h, a) \subseteq \mathbb{V}_{J,I}(h, a)$. Clearly, $\mathbb{V}_{I,I}(t, a) = \mathbb{V}_{J,I}(t, a)$ by definition. Furthermore there is at least one atom p such that $v_J(p) < v_I(p)$, i.e., $\mathbb{V}_{J,I}^-(h, a) < \mathbb{V}_{I,I}^-(h, a)$. So $\mathbb{V}_{J,I} \prec \mathbb{V}_{I,I}$. ■

Lemma 12. *An interpretation I is a stable model of F if and only if $\mathbb{V}_{I,I}$ is a fuzzy equilibrium model of F .*

Proof. (\Rightarrow) Suppose I is a stable model of F . As I is a model of F , by Corollary 3, $\mathbb{V}_{I,I}$ is a model of F . Next we show that there is no model V' of F such that $V' \prec \mathbb{V}_{I,I}$. Suppose, for the sake of contradiction, that there exists such V' . Define an interpretation J as $v_J(a) = V'^{-}(h, a)$ for all atoms a . Obviously $V' = \mathbb{V}_{J,I}$. As $V' = \mathbb{V}_{J,I}$ is a model of F , by Corollary 4, $J \models F^I$. Furthermore, as $V' = \mathbb{V}_{J,I} \prec \mathbb{V}_{I,I}$, by Lemma 11, $J < I$. Consequently, I is not a 1-stable model of F , which is a contradiction. So there does not exist such V' , from which we conclude that $\mathbb{V}_{I,I}$ is an h-minimal model of F . Clearly $\mathbb{V}_{I,I}(h, a) = \mathbb{V}_{I,I}(t, a)$ for all atoms a . So $\mathbb{V}_{I,I}$ is an equilibrium model of F .

(\Leftarrow) Suppose $\mathbb{V}_{I,I}$ is an equilibrium model of F . As $\mathbb{V}_{I,I}$ is a model of F , by Corollary 3, $I \models F$. Next we show that there is no $J <^\sigma I$ such that $J \models F^I$. Suppose, for the sake of contradiction, that there exists such J . Then by Corollary 4, the valuation $\mathbb{V}_{J,I}$ is a model of F . Furthermore, by Lemma 11, $\mathbb{V}_{J,I} \prec \mathbb{V}_{I,I}$. Consequently, $\mathbb{V}_{I,I}$ is not an h-minimal model of F , which contradicts that $\mathbb{V}_{I,I}$ is an equilibrium model of F . So there does not exist such J , from which we conclude that I is a stable model of F . ■

Lemma 13. *For any two valuations V and V' such that $V^-(w, a) = V'^-(w, a)$ ($w \in \{h, t\}$) for all atoms a , and any formula F containing no strong negation, V is a model of F iff V' is a model of F .*

Proof. We show by induction that $V^-(w, F) = V'^-(w, F)$, where $w \in \{h, t\}$.

- F is an atom p . $V^-(w, F) = V^-(w, p) = V'^-(w, p) = V^-(w, F)$.
- F is a numeric constant c . Clearly $V^-(w, F) = c = V'^-(w, F)$.
- F is $\neg G$. By I.H, $V^-(w, G) = V'^-(w, G)$.

$$V^-(w, F) = 1 - V^-(w, G) = 1 - V'^-(w, G) = V'^-(w, F).$$

- F is $G \odot H$ where \odot is \otimes or \oplus . By I.H, $V^-(w, G) = V'^-(w, G)$ and $V^-(w, H) = V'^-(w, H)$. So

$$\begin{aligned} V^-(w, F) &= V^-(w, G) \odot V^-(w, H) \\ &= V'^-(w, G) \odot V'^-(w, H) \\ &= V'^-(w, F). \end{aligned}$$

- F is $G \rightarrow H$. By I.H, $V^-(w, G) = V'^-(w, G)$ and $V^-(w, H) = V'^-(w, H)$. So

$$\begin{aligned} V^-(h, F) &= \min((V^-(h, G) \rightarrow V^-(h, H)), (V^-(t, G) \rightarrow V^-(t, H))) \\ &= \min((V'^-(h, G) \rightarrow V'^-(h, H)), (V'^-(t, G) \rightarrow V'^-(t, H))) \\ &= V'^-(h, F). \end{aligned}$$

And

$$\begin{aligned}
 V^-(t, F) &= (V^-(t, G) \rightarrow V^-(t, H)) \\
 &= (V'^-(t, G) \rightarrow V'^-(t, H)) \\
 &= V'^-(t, F).
 \end{aligned}$$

So $V^-(h, F) = V'^-(h, F)$ and thus $V^-(h, F) = 1$ if and only if $V'^-(h, F) = 1$, i.e., V is a model of F if and only if V' is a model of F . ■

Lemma 1. *Given a formula F containing no strong negation, any equilibrium model V of F satisfies $V^+(h, a) = V^+(t, a) = 1$ for all atoms a .*

Proof. Assume that V is an equilibrium model of F . It follows that $V^+(h, a) = V^+(t, a)$. Furthermore, for the sake of contradiction, assume that $V^+(h, a) = V^+(t, a) = v < 1$. Define V'' as $V''^-(w, a) = V^-(w, a)$, $V''^+(t, a) = V^+(t, a)$ and $V''^+(h, a) = v'$ where $v' \in (v, 1]$. Clearly $V'' \prec V$ and by Lemma 13, V'' is a model of F . So V is not an h-minimal model of F , which contradicts the assumption that V is an equilibrium model of F . Therefore, there does not exist such V . ■

Theorem 6. *Let F be a fuzzy propositional formula of σ that contains no strong negation.*

- (a) *A valuation V of σ is a fuzzy equilibrium model of F iff $V^-(h, p) = V^-(t, p)$, $V^+(h, p) = V^+(t, p) = 1$ for all atoms p in σ and \mathfrak{I}_V is a stable model of F relative to σ .*
- (b) *An interpretation I of σ is a stable model of F relative to σ iff $I = \mathfrak{I}_V$ for some fuzzy equilibrium model V of F .*

Proof. (a) (\Rightarrow) Suppose V is an equilibrium model of F . By Lemma 1, $V^+(h, a) = V^+(t, a) = 1$ for all atoms a . And by the definition of fuzzy equilibrium model, $V^-(h, a) = V^-(t, a)$. It can be seen that $\mathfrak{V}_{\mathfrak{I}_V, \mathfrak{I}_V} = V$. Since $\mathfrak{V}_{\mathfrak{I}_V, \mathfrak{I}_V}$ is an equilibrium model of F , by Lemma 12, \mathfrak{I}_V is a stable model of F relative to σ .

(\Leftarrow) Suppose $V^+(h, a) = V^+(t, a) = 1$, $V^-(h, a) = V^-(t, a)$ for all atoms a , and \mathfrak{I}_V is a stable model of F relative to σ . By Lemma 12, $\mathfrak{V}_{\mathfrak{I}_V, \mathfrak{I}_V}$ is an equilibrium model of F . Since $V^+(h, a) = V^+(t, a) = 1$, $V^-(h, a) = V^-(t, a)$ for all atoms a , it holds that $\mathfrak{V}_{\mathfrak{I}_V, \mathfrak{I}_V} = V$. So V is an equilibrium model of F .

(b) (\Rightarrow) Suppose I is a stable model of F relative to σ . By Lemma 12, $\mathfrak{V}_{I, I}$ is an equilibrium model of F . It can be seen that $I = \mathfrak{I}_{\mathfrak{V}_{I, I}}$.

(\Leftarrow) Take any fuzzy equilibrium model V of F and let $I = \mathsf{l}_V$. By the definition of fuzzy equilibrium model and Lemma 1, $V^-(h, a) = V^-(t, a)$ and $V^+(h, a) = V^+(t, a) = 1$ for all atoms a , which means $\mathsf{V}_{\mathsf{l}_V, \mathsf{l}_V} = V$. So $\mathsf{V}_{\mathsf{l}_V, \mathsf{l}_V}$ is an equilibrium model of F . By Lemma 12, l_V is a stable model, so I is a stable model. ■

A.8 Proof of Theorem 7

Lemma 14. *For any fuzzy formula F of signature σ that may contain strong negation and for any valuation V , it holds that $V^-(w, F) = \mathit{nneg}(V)^-(w, \mathit{nneg}(F))$.*

Proof. First we show by induction that $V^-(w, F) = \mathit{nneg}(V)^-(w, F')$, where F' is defined as in Section 6.2.3.

- F is an atom p in σ . Clear.
- F is $\sim p$, where p is an atom in σ . Then F' is np .

$$\begin{aligned} V^-(w, F) &= V^-(w, \sim p) = 1 - V^+(w, p) \\ &= \mathit{nneg}(V)^-(w, np) = \mathit{nneg}(V)^-(w, F'). \end{aligned}$$

- F is a numeric constant c . Clear.
- F is $\neg G$. By I.H., $V^-(w, G) = \mathit{nneg}(V)^-(w, G')$.

$$\begin{aligned} V^-(w, F) &= V^-(w, \neg G) \\ &= 1 - V^-(w, G) \\ &= 1 - \mathit{nneg}(V)^-(w, G') \\ &= \mathit{nneg}(V)^-(w, \neg G') \\ &= \mathit{nneg}(V)^-(w, F'). \end{aligned}$$

- F is $G \odot H$, where \odot is \otimes or \oplus . By I.H., $V^-(w, G) = \mathit{nneg}(V)^-(w, G')$ and $V^-(w, H) = \mathit{nneg}(V)^-(w, H')$.

$$\begin{aligned} V^-(w, F) &= V^-(w, G \odot H) \\ &= \odot(V^-(w, G), V^-(w, H)) \\ &= \odot(\mathit{nneg}(V)^-(w, G'), \mathit{nneg}(V)^-(w, H')) \\ &= \mathit{nneg}(V)^-(w, G' \odot H') \\ &= \mathit{nneg}(V)^-(w, F'). \end{aligned}$$

- F is $G \rightarrow H$. By I.H., $V^-(w, G) = \text{nneg}(V)^-(w, G')$ and $V^-(w, H) = \text{nneg}(V)^-(w, H')$.

$$\begin{aligned}
 V^-(h, F) &= V^-(h, G \rightarrow H) \\
 &= \min(\rightarrow(V^-(h, G), V^-(h, H)), \rightarrow(V^-(t, G), V^-(t, H))) \\
 &= \min(\rightarrow(\text{nneg}(V)^-(h, G'), \text{nneg}(V)^-(h, H')), \\
 &\quad \rightarrow(\text{nneg}(V)^-(t, G'), \text{nneg}(V)^-(t, H'))) \\
 &= \text{nneg}(V)^-(h, G' \rightarrow H') \\
 &= \text{nneg}(V)^-(h, F').
 \end{aligned}$$

And

$$\begin{aligned}
 V^-(t, F) &= V^-(t, G \rightarrow H) \\
 &= \rightarrow(V^-(t, G), V^-(t, H)) \\
 &= \rightarrow(\text{nneg}(V)^-(t, G'), \text{nneg}(V)^-(t, H')) \\
 &= \text{nneg}(V)^-(t, F').
 \end{aligned}$$

Now notice that, for any valuation V , it must be the case that for all atoms $p \in \sigma$, $V^-(w, p) \leq V^+(w, p)$, i.e, $V^-(w, p) + 1 - V^+(w, p) \leq 1$. It follows that

$$\text{nneg}(V)^-(w, p) + \text{nneg}(V)^-(w, np) \leq 1.$$

Therefore, for all atoms p ,

$$\begin{aligned}
 \text{nneg}(V)^-(w, \neg_s(p \otimes_l np)) &= 1 - \text{nneg}(V)^-(t, (p \otimes_l np)) \\
 &= 1 - \otimes_l(\text{nneg}(V)^-(t, p), \text{nneg}(V)^-(t, np)) \\
 &= 1 - \max(\text{nneg}(V)^-(t, p) + \text{nneg}(V)^-(t, np) - 1, 0) \\
 &= 1 - 0 = 1.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 V^-(w, F) &= \text{nneg}(V)^-(w, F') \\
 &= \otimes_m(\text{nneg}(V)^-(w, F'), 1) \\
 &= \otimes_m(\text{nneg}(V)^-(w, F'), \text{nneg}(V)^-(w, \otimes_m \neg_s(p \otimes_l np))) \\
 &= \text{nneg}(V)^-(w, F' \otimes_m \otimes_m \neg_s(p \otimes_l np)) \\
 &= \text{nneg}(V)^-(w, \text{nneg}(F)).
 \end{aligned}$$

■

Corollary 5. *For any fuzzy formula F that may contain strong negation, a valuation V is a model of F iff $nneg(V)$ is a model of $nneg(F)$.*

Proof. By Lemma 14, $V^-(h, F) = nneg(V)^-(h, nneg(F))$. So $V^-(h, F) = 1$ iff $nneg(V)^-(h, nneg(F)) = 1$, i.e., V is a model of F iff $nneg(V)$ is a model of $nneg(F)$. ■

Lemma 15. *for all atoms $a \in \sigma$, $V(h, a) = V(t, a)$ iff*

$$nneg(V)(h, a) = nneg(V)(t, a) \text{ and } nneg(V)(h, na) = nneg(V)(t, na).$$

Proof. For all atoms $a \in \sigma$,

$$\begin{aligned} & V(h, a) = V(t, a) \\ \text{iff } & V^-(h, a) = V^-(t, a) \text{ and } V^+(h, a) = V^+(t, a) \\ \text{iff } & nneg(V)(h, a) = nneg(V)(t, a) \text{ and } 1 - V^+(h, a) = 1 - V^+(t, a) \\ \text{iff } & nneg(V)(h, a) = nneg(V)(t, a) \text{ and } nneg(V)(h, na) = nneg(V)(t, na). \end{aligned}$$

■

Lemma 16. *For two valuations V and V_1 of signature σ , it holds that $V_1 \prec V$ iff $nneg(V_1) \prec nneg(V)$.*

Proof.

- $V_1 \prec V$
- iff for all atoms p (from σ), $V(t, p) = V_1(t, p)$, $V(h, p) \subseteq V_1(h, p)$ and there exists an atom a (from σ) such that $V(h, a) \subset V_1(h, a)$
- iff for all atoms p , $V^-(t, p) = V_1^-(t, p)$, $V^+(t, p) = V_1^+(t, p)$,
 $V_1^-(h, p) \leq V^-(h, p)$, $V_1^+(h, p) \geq V^+(h, p)$, and
 there exists an atom a such that $V_1^-(h, a) < V^-(h, a)$ or $V_1^+(h, a) > V^+(h, a)$
- iff for all atoms p , $V^-(t, p) = V_1^-(t, p)$, $V^+(t, p) = V_1^+(t, p)$,
 $V_1^-(h, p) \leq V^-(h, p)$, $V_1^+(h, p) \geq V^+(h, p)$, and
 there exists an atom a such that $V_1^-(h, a) < V^-(h, a)$ or
 $1 - V_1^+(h, a) < 1 - V^+(h, a)$
- iff for all atoms p , $V^-(t, p) = V_1^-(t, p)$, $V^+(t, p) = V_1^+(t, p)$,
 $V_1^-(h, p) \leq V^-(h, p)$, $1 - V_1^+(h, p) \leq 1 - V^+(h, p)$, and
 there exists an atom a such that $V_1^-(h, a) < V^-(h, a)$ or
 $nneg(V_1)^-(h, na) < nneg(V)^-(h, na)$
- iff for all atoms p and np , $nneg(V)(t, p) = nneg(V_1)(t, p)$, $nneg(V)(t, np) = nneg(V_1)(t, np)$,
 $nneg(V)(h, p) \subseteq nneg(V_1)(h, p)$, $nneg(V)(h, np) \subseteq nneg(V_1)(h, np)$, and
 there exists an atom a or na such that $nneg(V)(h, a) \subset nneg(V_1)(h, a)$ or
 $nneg(V)(h, na) \subset nneg(V_1)(h, na)$
- iff $nneg(V_1) \prec nneg(V)$.

■

Proposition 3. *For any fuzzy formula F that may contain strong negation, a valuation V is an equilibrium model of F iff $nneg(V)$ is an equilibrium model of $nneg(F)$.*

Proof. Let σ be the underlying signature of F , and let σ' be the extended signature $\sigma \cup \{np \mid p \in \sigma\}$.

(\Rightarrow) Suppose V is an equilibrium model of F . Then $V(h, p) = V(t, p)$ for all atoms $p \in \sigma$ and V is a model of F . By Lemma 15, $nneg(V)(h, a) = nneg(V)(t, a)$ for all atoms $a \in \sigma'$, and by Corollary 5, $nneg(V)$ is a model of $nneg(F)$.

Next we show that there is no $V_1 \prec nneg(V)$ such that V_1 is a model of $nneg(F)$. Suppose, for the sake of contradiction, that there exists such V_1 . We construct V' as

$$V'(w, p) = [V_1^-(w, p), 1 - V_1^-(w, np)]$$

for all atoms $p \in \sigma$. It is clear that $nneg(V') = V_1$. We will check that

- (i) V' is a valid N5 valuation,
- (ii) $V' \prec V$, and
- (iii) V' is a model of F .

These claims contradicts the assumption that V is an equilibrium model of F . Consequently, we conclude that $nneg(V)$ is an equilibrium model of F' once we establish these claims.

To check (i), we need to show that $0 \leq V_1^-(w, p) \leq 1 - V_1^-(w, np) \leq 1$ and $V'(t, p) \subseteq V'(h, p)$ for all atoms $p \in \sigma$. Together they are equivalent to checking

$$0 \leq V_1^-(h, p) \leq V_1^-(t, p) \leq 1 - V_1^-(t, np) \leq 1 - V_1^-(h, np) \leq 1$$

The first and the last inequalities are obvious. The second and the fourth inequalities are clear from the fact that V_1 is a valid valuation. To show the third inequality, since V is a valid valuation,

$$V^-(t, p) \leq V^+(t, p)$$

which is equivalent to

$$nneg(V)^-(t, p) \leq 1 - nneg(V)^-(t, np). \tag{1}$$

Since $V_1 \prec nneg(V)$, we have $V_1(t, a) = nneg(V)(t, a)$ for all atoms $a \in \sigma'$, so (1) is equivalent to

$$V_1^-(t, p) \leq 1 - V_1^-(t, np).$$

So (i) is verified.

To check claim (ii), notice $nneg(V') \prec nneg(V)$ since $V_1 \prec nneg(V)$ and $V_1 = nneg(V')$. Then the claim follows by Lemma 16.

To check claim (iii), notice $nneg(V')$ is a model of $nneg(F)$ since V_1 is a model of $nneg(F)$, and $V_1 = nneg(V')$. Then the claim follows by Corollary 5.

(\Leftarrow) Suppose $nneg(V)$ is an equilibrium model of $nneg(F)$. Then $nneg(V)(h, a) = nneg(V)(t, a)$ for all atoms $a \in \sigma'$ and $nneg(V)$ is a model of $nneg(F)$. By Lemma 15, $V(h, a) = V(t, a)$ for all atoms $a \in \sigma'$ and by Corollary 5, V is a model of F . Next we show that there is no $V_1 \prec V$ such that V_1 is a model of F . Suppose, for the sake of contradiction, that there exists such V_1 . Then by Lemma 16, $nneg(V_1) \prec nneg(V)$ and by Corollary 5, $nneg(V_1)$ is a model of $nneg(F)$, which contradicts that $nneg(V)$ is an equilibrium model of $nneg(F)$. Consequently, we conclude that V is an equilibrium model of F . ■

Theorem 7. *For any fuzzy formula F of signature σ that may contain strong negation,*

- (a) *A valuation V of σ is a fuzzy equilibrium model of F iff $V(h, p) = V(t, p)$ for all atoms p in σ and $\mathsf{l}_{nneg(V)}$ is a stable model of $nneg(F)$ relative to $\sigma \cup \{np \mid p \in \sigma\}$.*
- (b) *An interpretation I of $\sigma \cup \{np \mid p \in \sigma\}$ is a stable model of $nneg(F)$ relative to $\sigma \cup \{np \mid p \in \sigma\}$ iff $I = \mathsf{l}_{nneg(V)}$ for some fuzzy equilibrium model V of F .*

Proof. (a) (\Rightarrow) Suppose V is an equilibrium model of F . By the definition of an equilibrium model, $V(h, a) = V(t, a)$ for all atoms a . By Proposition 3, $nneg(V)$ is an equilibrium model of $nneg(F)$. By Theorem 6, $\mathsf{l}_{nneg(V)}$ in the sense of Theorem 6 is a stable model of $nneg(F)$ relative to $\sigma \cup \{np \mid p \in \sigma\}$.

(\Leftarrow) Suppose $\mathsf{l}_{nneg(V)}$ in the sense of Theorem 6 is a stable model of $nneg(F)$ relative to $\sigma \cup \{np \mid p \in \sigma\}$. By Theorem 6, $nneg(V)$ is an equilibrium model of $nneg(F)$. By Proposition 3, V is an equilibrium model of F .

(b) (\Rightarrow) Suppose I is a stable model of $nneg(F)$ relative to $\sigma \cup \{np \mid p \in \sigma\}$. Then $I \models \bigotimes_{m \in \sigma} \neg_s(p \otimes_l np)$. It follows that, for all atoms $p \in \sigma$, $v_I(p) + v_I(np) \leq 1$ and thus $v_I(p) \leq 1 - v_I(np)$. Construct the valuation V of σ by defining $V(w, p) = [v_I(p), 1 - v_I(np)]$. Clearly, I can be viewed as $\mathsf{l}_{nneg(V)}$. By Theorem 6, $nneg(V)$ is an equilibrium model of $nneg(F)$. By Proposition 3, V is an equilibrium model of F .

(\Leftarrow) Take any fuzzy equilibrium model V of F . By Proposition 3, $nneg(V)$ is an equilibrium model of $nneg(F)$. By Theorem 6, $\mathsf{l}_{nneg(V)}$ is a stable model of $nneg(F)$.

■

A.9 Proof of Theorem 8

Theorem 8. *For any fuzzy formulas F and G , I is a stable model of $F \otimes \neg G$ (relative to \mathbf{p}) if and only if I is a stable model of F (relative to \mathbf{p}) and $I \models \neg G$.*

Proof.

(\Rightarrow) Suppose I is a 1-stable model of $F \otimes \neg G$ relative to \mathbf{p} . Then $I \models_1 F \otimes \neg G$ and there is no $J <^{\mathbf{p}} I$ such that $J \models_1 F^I \otimes (\neg G)^I$. By Lemma 6, $I \models_1 F$ and $I \models_1 \neg G$. Note that $v_I(\neg G) = 1$. Further, it can be seen that there is no $J <^{\mathbf{p}} I$ such that $J \models_1 F^I$, since otherwise this J must satisfy $J <^{\mathbf{p}} I$ and $J \models_1 F^I \otimes v_I(\neg G)$ (i.e.,

$J \models_1 F^I \otimes (\neg G)^I$), which contradicts that I is 1-stable model of $F \otimes \neg G$ relative to \mathbf{p} . We conclude that I is a 1-stable model of F relative to \mathbf{p} and $I \models_1 \neg G$.

(\Leftarrow) Suppose I is a 1-stable model of F relative to \mathbf{p} and $I \models_1 \neg G$. Then $I \models_1 F$ and $I \models_1 \neg G$. By Lemma 6, $I \models_1 F \otimes \neg G$.¹² Next we show that there is no $J <^{\mathbf{P}} I$ such that $J \models_1 (F \otimes \neg G)^I = F^I \otimes (\neg G)^I$. Suppose, to the contrary, that there exists such J . Then by Lemma 6, $J \models_1 F^I$. This, together with the fact $J <^{\mathbf{P}} I$, contradicts that I is a 1-stable model of F relative to \mathbf{p} . So there does not exist such J , and we conclude that I is a 1-stable model of $F \otimes \neg G$ relative to \mathbf{p} . ■

A.10 Proof of Theorem 9

Proposition 4. *For any fuzzy interpretation I and any set \mathbf{p} of fuzzy atoms, $I \models \mathbf{p}^{\text{ch}}$.*

Proof. Suppose $\mathbf{p} = (p_1, \dots, p_n)$.

$$\begin{aligned} v_I(\mathbf{p}^{\text{ch}}) &= v_I(\{p_1\}^{\text{ch}} \otimes \dots \otimes \{p_n\}^{\text{ch}}) \\ &= v_I((p_1 \oplus_l \neg_s p_1) \otimes \dots \otimes (p_n \oplus_l \neg_s p_n)) \\ &= \min(v_I(p_1) + 1 - v_I(p_1), 1) \otimes \dots \otimes \min(v_I(p_n) + 1 - v_I(p_n), 1) \\ &= 1. \end{aligned}$$

■

Theorem 9.

- (a) *For any real number $y \in [0, 1]$, if I is a y -stable model of F relative to $\mathbf{p} \cup \mathbf{q}$, then I is a y -stable model of F relative to \mathbf{p} .*
- (b) *I is a 1-stable model of F relative to \mathbf{p} iff I is a 1-stable model of $F \otimes \mathbf{q}^{\text{ch}}$ relative to $\mathbf{p} \cup \mathbf{q}$.*

Proof. (a) Suppose I is a y -stable model of F relative to $\mathbf{p} \cup \mathbf{q}$. Then clearly $I \models_y F$. Next we show that there is no $J <^{\mathbf{P}} I$ such that $J \models_y F^I$. Suppose, for the sake of contradiction, that there exists such J . Since $v_J(q) = v_I(q)$ for all $q \in \mathbf{q}$, J must satisfy $J <^{\mathbf{P}\mathbf{q}} I$ and $J \models_y F^I$, which contradicts the assumption that I is a y -stable model of F relative to $\mathbf{p} \cup \mathbf{q}$. So such J does not exist, and we conclude that I is a y -stable model of F relative to \mathbf{p} .

¹²This does not hold if the threshold considered is not 1. For example, suppose $v_I(F) = 0.5$ and $v_I(G) = 0.5$, and consider \otimes_l as the fuzzy conjunction. Clearly $I \models_{0.5} F$ and $I \models_{0.5} G$ but $I \not\models_{0.5} F \otimes_l G$.

(b) (\Rightarrow) Suppose I is a 1-stable model of F relative to \mathbf{p} . Clearly $I \models F$. By Proposition 4, $I \models \mathbf{q}^{\text{ch}}$, and by Lemma 6, $I \models F \otimes \mathbf{q}^{\text{ch}}$.¹³

Next we show that there is no $J <^{\mathbf{p}\mathbf{q}} I$ such that $J \models (F \otimes \mathbf{q}^{\text{ch}})^I$. Suppose, for the sake of contradiction, that there exists such J . Since $J \models (F \otimes \mathbf{q}^{\text{ch}})^I$, i.e.,

$$J \models F^I \otimes (q_1 \oplus_l (\neg_s q_1)^I) \otimes \dots \otimes (q_n \oplus_l (\neg_s q_n)^I),$$

by Lemma 6, it follows that $J \models (q_k \oplus_l v_I(\neg_s q_k))$ for each $k = 1, \dots, n$, which means that $v_J(q_k) \geq v_I(q_k)$. On the other hand, since $J <^{\mathbf{p}\mathbf{q}} I$, we have $v_J(q_k) \leq v_I(q_k)$. So we conclude $v_J(q_k) = v_I(q_k)$.¹⁴ Together with the assumption that $J <^{\mathbf{p}\mathbf{q}} I$, this implies that $J <^{\mathbf{p}} I$. From $J \models (F \otimes \mathbf{q}^{\text{ch}})^I$, it follows $J \models F^I$, which contradicts that I is a stable model of F relative to \mathbf{p} . So such J does not exist, and we conclude that I is a stable model of F relative to $\mathbf{p} \cup \mathbf{q}$.

(\Leftarrow) Suppose I is a 1-stable model of $F \otimes \mathbf{q}^{\text{ch}}$ relative to $\mathbf{p} \cup \mathbf{q}$. Then $I \models F \otimes \mathbf{q}^{\text{ch}}$. By Lemma 6, $I \models F$. Next we show that there is no $J <^{\mathbf{p}} I$ such that $J \models F^I$. Suppose, for the sake of contradiction, that there exists such J .

First, it is easy to conclude $J <^{\mathbf{p}\mathbf{q}} I$ from the fact that $J <^{\mathbf{p}} I$ since J and I agree on \mathbf{q} . Second, by Proposition 4, $J \models \mathbf{q}^{\text{ch}}$. Since J and I agree on \mathbf{q} , it follows that $J \models (\mathbf{q}^{\text{ch}})^I$. Since $J \models F^I$, by Lemma 6, $J \models F^I \otimes (\mathbf{q}^{\text{ch}})^I$, or equivalently, $J \models (F \otimes \mathbf{q}^{\text{ch}})^I$. This, together with the fact that $J <^{\mathbf{p}\mathbf{q}} I$, contradicts that I is a 1-stable model of $F \otimes \mathbf{q}^{\text{ch}}$ relative to $\mathbf{p} \cup \mathbf{q}$. So such J does not exist, and we conclude that I is a 1-stable model of F relative to \mathbf{p} . ■

¹³It is not necessary to have the threshold $y = 1$ for this to hold. In general, suppose $I \models_y F$. Since $I \models_1 \mathbf{p}^{\text{ch}}$, by the property of fuzzy conjunction ($\otimes(x, 1) = x$), $I \models_y F \otimes \mathbf{p}^{\text{ch}}$.

¹⁴This cannot be concluded if we have $J \models_y (q_k \oplus_l v_I(\neg_s q_k))$, instead of $J \models_1 (q_k \oplus_l v_I(\neg_s q_k))$. So this direction of the theorem does not hold for an arbitrary threshold.