On the Stable Model Semantics for Intensional Functions

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How Do We Represent NonBoolean Fluents in Answer Set Programming?

NonBoolean fluents: the location of an object, the color of a ball, ...

In classical logic, nonBoolean fluents can be naturally described by functions.

- $loc(b) = table; \quad color(b) = red$

This is not the case with the traditional stable model semantics.

- "Minimal belief with negation as failure" is related to the minimality condition for predicates but has nothing to do with functions.
  - (okay)$WaterLevel(t+1, l) \leftarrow WaterLevel(t, l),\ not\ \sim WaterLevel(t+1, l)$.
  - (wrong)$WaterLevel(t+1) = l \leftarrow WaterLevel(t) = l,\ not\ WaterLevel(t+1) \neq l$.
- Stable models are limited to Herbrand models.
  - $Loc(B) = Loc(B_1)$ is always false
- Grounding generates a large number of instances as the domain gets larger.

Recent Developments

There are two recent lines of research to enhance ASP with functions.

- Integrating ASP with CSP / SMT: to improve the computational efficiency by addressing the grounding problem
  - [Gebser, Ostrowski, and Schaub, ICLP 2009]
  - [Balduccini, ASPOCP 2009]
  - [Janhunen, Liu, Niemelä, KR 2012]

  But
  $WaterLevel(t{+}1) = l \leftarrow WaterLevel(t) = l,\ not\ WaterLevel(t{+}1) \neq l.$
  does not work.
- Intensional Functions: to enrich the modeling language
  - [Cabalar, TPLP 2011]
  - [Lifschitz, KR 2012]
  - [Bartholomew and Lee, KR 2012]
  - [Balduccini, Correct Reasoning 2012]

Answer Set Programming Modulo Theories (ASPMT)

- ASPMT [Bartholomew and Lee, IJCAI 2013] attempts to merge these two categories.
- ASPMT tightly integrates ASP and SMT:

Monotonic	Nonmonotonic
FOL	Functional Stable Model Semantics [Bartholomew and Lee, KR 2012]
SMT	ASP Modulo Theories [Bartholomew and Lee, IJCAI 2013]
SAT	Traditional ASP

- The Bartholomew-Lee semantics can be computed by SMT solvers.

Outcome

The definitions of intensional functions are not presented in similar fashion:

	Bartholomew-Lee Semantics	Cabalar Semantics	Balduccini Semantics
Second-order Logic	[Bartholomew and Lee 2012]	This paper	
Variant of HT Logic	This paper, [del Cerro et al. 2013]	[Cabalar 2011]	
Reduct	This paper	This paper	[Balduccini 2012]

- The Bartholomew-Lee and Cabalar semantics coincide on two large classes of formula: **c**-plain formulas and tight head-**c**-plain formulas.
- The Balduccini semantics is a special case of the Cabalar semantics.

Outline

- Grounding and Reduct Reformulation of Bartholomew-Lee Semantics
- Comparing the Bartholomew-Lee Semantics and the Cabalar Semantics
- Comparing the Cabalar Semantics and the Balduccini Semantics

Grounding and Reduct Reformulation of Bartholomew-Lee Semantics

Functional Stable Model Semantics (FSM) [Bartholomew and Lee, 2012]

Allows for assigning default values to non-Herbrand functions, which is useful for expressing inertia and default behaviors of systems.

Leaking Container Example

$$\begin{array}{rcl} \{Amount_1 = x\} & \leftarrow & Amount_0 = x+1 \\ Amount_1 = 10 & \leftarrow & FillUp\ . \end{array}$$

$\{Amount_1 = x\}$ is a choice rule standing for $Amount_1 = x \vee \neg(Amount_1 = x)$

- $I_1 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = 5\}$:
  I_1 is a stable model of F (relative to $Amount_1$) as well as a model.
- $I_2 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = 8\}$:
  I_2 is a model of F but not a stable model.
- $I_3 = \{FillUp = \text{TRUE}, Amount_0 = 6, Amount_1 = 10\}$:
  I_3 is a model of F as well as a stable model of F.

FSM in Terms of SOL

FSM is a proper extension of First-Order Stable Model Semantics from [Ferraris et al., AIJ 2011].
$\mathbf{c}$ is a list of predicate and function constants called intensional.

$\mathbf{u}$ is a list of predicate and function variables corresponding to $\mathbf{c}$.

$\mathrm{SM}[F;\ \mathbf{c}]$ is defined as

$$F \wedge \neg\exists\mathbf{u}(\mathbf{u} < \mathbf{c} \wedge F^*(\mathbf{u}))$$

where $F^*(\mathbf{u})$ is defined as:

- when F is an atomic formula, F^* is $F(\mathbf{u}) \wedge F$;
- $(G \wedge H)^* = G^* \wedge H^*$; $\quad (G \vee H)^* = G^* \vee H^*$;
  $(G \to H)^* = (G^* \to H^*) \wedge (G \to H)$;
- $(\forall x G)^* = \forall x G^*$; $\quad (\exists x F)^* = \exists x F^*$.

Infinitary Ground Formula w.r.t. an Interpretation

Before giving the reduct-based reformulation of the Bartholomew-Lee semantics, we define the notion of grounding.

Since the universe may be infinite, grounding a first-order sentence F relative to an interpretation I (denoted $gr_I[F]$) may introduce infinite conjunctions and disjunctions. We adapt this idea from [Trusczyinski Correct Reasoning 2012].

Leaking Container Example. $gr_I[F]$ is

$$\begin{array}{rcl} \{Amount_1 = 0\} & \leftarrow & Amount_0 = 0+1 \\ \{Amount_1 = 1\} & \leftarrow & Amount_0 = 1+1 \\ & \dots & \\ Amount_1 = 10 & \leftarrow & FillUp \end{array}$$

Reduct-based Definition of FSM

For any two interpretations I, J of the same signature and any list $\mathbf{c}$ of distinct predicate and function constants, we write $J <^{\mathbf{c}} I$ if

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants p in $\mathbf{c}$; and
- J and I do not agree on $\mathbf{c}$.

The reduct $F^{\underline{I}}$ of an infinitary ground formula F relative to an interpretation I is the formula obtained from F by replacing every "maximal subformula" that is not satisfied by I with $\bot$.

Theorem (1)

I is a stable model of F as defined in [Bartholomew and Lee 2012] iff

- *I satisfies F, and*
- *every interpretation J such that $J <^{\mathbf{c}} I$ does not satisfy $(gr_I[F])^{\underline{I}}$.*

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Leaking Container Example. Let $I_1 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = 5\}$. Then $gr_{I_1}[F]$ is

$$\begin{array}{ccc} \{Amount_1 = 0\} & \leftarrow & Amount_0 = 0+1 \\ \{Amount_1 = 1\} & \leftarrow & Amount_0 = 1+1 \\ & \dots & \\ Amount_1 = 10 & \leftarrow & FillUp \end{array}$$

The reduct $gr_{I_1}[F]^{\underline{I_1}}$ is

$$\begin{array}{ccc} \bot \vee \neg\bot & \leftarrow & \bot \\ & \dots & \\ Amount_1 = 5 \vee \bot & \leftarrow & Amount_0 = 5+1 \\ & \dots & \\ \bot & \leftarrow & \bot \end{array}$$

Any J_1 such that $J_1 <^{Amount_1} I_1$ (J_1 disagrees with I_1 on $Amount_1$) does not satisfy $gr_{I_1}[F]^{\underline{I_1}}$.

E.g., $J_1 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = 3\}$ does not satisfy $gr_{I_1}[F]^{\underline{I_1}}$.

Reduct-based Definition of FSM

Leaking Container Example. Let $I_2 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = 8\}$. Then $gr_{I_2}[F]$ is

$$\begin{array}{ccc} \{Amount_1 = 0\} & \leftarrow & Amount_0 = 0+1 \\ \{Amount_1 = 1\} & \leftarrow & Amount_0 = 1+1 \\ & \dots & \\ Amount_1 = 10 & \leftarrow & FillUp \end{array}$$

The reduct $gr_{I_2}[F]^{\underline{I_2}}$ is

$$\begin{array}{ccc} \bot \vee \neg\bot & \leftarrow & \bot \\ & \dots & \\ \bot \vee \neg\bot & \leftarrow & Amount_0 = 5+1 \\ & \dots & \\ Amount_1 = 8 \vee \bot & \leftarrow & \bot \\ \bot & \leftarrow & \bot \end{array}$$

Now we can find a $J_2 <^{Amount_1} I_2$ (J_2 disagrees with I_2 on $Amount_1$) that satisfies $gr_{I_2}[F]^{\underline{I_2}}$. For instance,
$J_2 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = 7\}$ satisfies $gr_{I_2}[F]^{\underline{I_2}}$.

Comparing the Bartholomew-Lee Semantics and the Cabalar Semantics

Partial Interpretations

The Cabalar semantics is defined in terms of partial interpretations.
A partial interpretation I of signature σ consists of

- a non-empty set $|I|$, called the universe of I;
- for every function constant f of σ of arity n, a function f^I from $(|I| \cup \{u\})^n$ to $|I| \cup \{u\}$, where u is not in $|I|$ (standing for undefined);
- for every predicate constant p of σ of arity n, a function p^I from $(|I| \cup \{u\})^n$ to $\{\text{TRUE}, \text{FALSE}\}$.

For each term $f(t_1, \dots, t_n)$,

$$f(t_1, \dots, t_n)^I = \begin{cases} u & \text{if } t_i^I = u \text{ for some } i \in \{1, \dots, n\} \\ f^I(t_1^I, \dots, t_n^I) & \text{otherwise.} \end{cases}$$

Partial Satisfaction

The satisfaction relation $\models_{\mathrm{p}}$ between a partial interpretation I and a first-order formula F is the same as the one for first-order logic except for the following base cases:

- For each atomic formula $p(t_1, \dots, t_n)$,

$$p(t_1, \dots, t_n)^I = \begin{cases} \text{FALSE} & \text{if } t_i^I = u \text{ for some } i \in \{1, \dots, n\} \\ p^I(t_1^I, \dots, t_n^I) & \text{otherwise.} \end{cases}$$

- For each atomic formula $t_1 = t_2$,

$$(t_1 = t_2)^I = \begin{cases} \text{TRUE} & t_1^I \neq u,\ t_2^I \neq u, \text{ and } t_1^I = t_2^I \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

We say $I \models_{\mathrm{p}} F$ if $F^I = \text{TRUE}$.
Under a partial interpretation,

- $t = t$ is not necessarily true: $I \not\models_{\mathrm{p}} t = t$ iff $t^I = u$.
- $\neg(t_1 = t_2)$, is true if one or both of t_1^I and t_2^I are mapped to u.

Functional Equilibrium Models [Cabalar 2011]

Given any two partial interpretations J and I of the same signature σ, and a set of constants $\mathbf{c}$, we write $J \preceq^{\mathbf{c}} I$ if

- J and I have the same universe and agree on all constants not in $\mathbf{c}$;
- $p^J \subseteq p^I$ for all predicate constants in $\mathbf{c}$; and
- $f^J(\xi) = u$ or $f^J(\xi) = f^I(\xi)$ for all function constants in $\mathbf{c}$ and all lists ξ of elements in the universe.

We write $J \prec^{\mathbf{c}} I$ if $J \preceq^{\mathbf{c}} I$ but not $I \preceq^{\mathbf{c}} J$. Note the similarity between the minimality of functions here and the minimality of predicates in ASP:

$$\begin{array}{lll} \emptyset & <^p & \{p\} \\ \{f = u\} & \prec^f & \{f = 1\} \end{array}$$

Reformulation in Terms of Grounding and Reduct

The Cabalar semantics is defined in terms of a varaint of HT-logic with partial satisfaction where models are called partial equilibrium models.

The Cabalar semantics can also be reformulated in terms of grounding and reduct similar to the reformulation of FSM.

Theorem (4)

Let F be a first-order sentence of signature σ and let $\mathbf{c}$ be a list of intensional constants. For any partial interpretation I of σ, $\langle I, I\rangle$ is a partial equilibrium model of F as defined in [Cabalar 2011] iff

- $I \models_{\mathrm{p}} F$, *and*
- *for every partial interpretation J of σ such that $J \prec^{\mathbf{c}} I$, we have $J \not\models_{\mathrm{p}} gr_I[F]^{\underline{I}}$.*

Reformulation in Terms of Grounding and Reduct

Take F to be the Leaking Container Example

$$\begin{array}{rcl} Amount_1 = x \vee \neg Amount_1 = x & \leftarrow & Amount_0 = x+1 \\ Amount_1 = 10 & \leftarrow & FillUp\ . \end{array}$$

and $I_1 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = 5\}$ so the reduct $gr_{I_1}[F]^{\underline{I_1}}$ is

$$\begin{array}{rcl} \bot \vee \neg\bot & \leftarrow & \bot \\ & \cdots & \\ Amount_1 = 5 \vee \bot & \leftarrow & Amount_0 = 5+1 \\ & \cdots & \\ \bot & \leftarrow & \bot \end{array}$$

The only J_1 such that $J_1 \prec^{Amount_1} I_1$ is $J_1 = \{FillUp = \text{FALSE}, Amount_0 = 6, Amount_1 = u\}$ and this does not satisfy $gr_{I_1}[F]^{\underline{I_1}}$ so I_1 is an equilibrium model.

Coincidence on $\mathbf{c}$-plain Formulas

A partial interpretation I is called total if I does not map any function constant to u. Obviously, a total interpretation can be identified with the classical interpretation.

For any function constant f, a first-order formula F is called f-plain if each atomic formula in F

- does not contain f, or
- is of the form $f(\mathbf{t}) = t_1$ where $\mathbf{t}$ is a list of terms not containing f, and t_1 is a term not containing f.

For example,

- $f = 1$ and $f(g) = g$ are f-plain.
- $p(f)$, $g(f) = 1$, and $1 = f$ are not f-plain.

Coincidence on **c**-plain Formulas

For a list **c** of predicate and function constants, we say that F is **c**-plain if F is f-plain for each function constant f in **c**.

- $f = 1 \wedge g = 2$ is (f, g)-plain
- $f = g$ and $g(f) = 1$ are not (f, g)-plain.

Coincidence on **c**-plain Formulas

Theorem (5)

*For any **c**-plain sentence F of signature σ, any list **c** of intensional constants, and any total interpretation I of σ satisfying $\exists xy(x \neq y)$, I is a stable model of F according to [Bartholomew and Lee 2012] iff I is a partial equilibrium model of F.*

The Leaking Container example demonstrated this theorem. However, the result does not hold for non **c**-plain formulas:

F is $f = g$, which is not (f, g)-plain. With the universe $\{1, 2\}$ and interpretation $I_1 = \{f = 1, g = 1\}$, the reduct $gr_{I_1}[F]^{\underline{I_1}}$ is $f = g$.

- I_1 is not a stable model of F with respect to f, g. Take $J_1 = \{f = 2, g = 2\}$. $J_1 <^{(f,g)} I_1$ and J_1 satisfies $gr_{I_1}[F]^{\underline{I_1}}$.
- I_1 is a partial equilibrium model of F. There is no J_2 such that $J_2 \prec^{(f,g)} I_1$ that satisfies $gr_{I_1}[F]^{\underline{I_1}}$.

Coincidence on Tight Head-**c**-plain Formula

We say that a formula is head-**c**-plain if the "head" of every rule is **c**-plain. We say that F is tight (on **c**) if the dependency graph of F (relative to **c**) is acyclic.

- $h\!=\!1 \leftarrow f(g)\!=\!1$ is tight.
- $g\!=\!1 \leftarrow f(g)\!=\!1$ is not tight.

Note that neither tight head-**c**-plain nor **c**-plain encompasses the other.

- $h\!=\!1 \leftarrow f(g)\!=\!1$ is head-(f,g,h)-plain but not (f,g,h)-plain.
- $f = 1 \leftarrow f = 1$ is not tight head-f-plain but it is f-plain.

Coincidence on Tight Head-**c**-plain Formula

Theorem (6)

*For any head-**c**-plain sentence F of signature σ that is tight on **c**, and any total interpretation I of σ satisfying $\exists xy(x \neq y)$, I is a stable model of F iff I is a partial equilibrium model of F.*

F is $f(1) = 1 \wedge f(2) = 1 \wedge (f(g) = 1 \rightarrow g = 1)$ which is not tight head-(f, g)-plain. With universe $\{1, 2\}$ and interpretation $I = \{f(1) = 1, f(2) = 1, g = 1\}$, the reduct $gr_I[F]^{\underline{I}}$ is exactly F.

- I is a stable model of F. For instance, $J_1 = \{f(1) = 1, f(2) = 1, g = 2\}$ is an interpretation such that $J_1 <^{(f,g)} I$ but J_1 doesn't satisfy $gr_I[F]^{\underline{I}}$.

- I is not a partial equilibrium model of F. $J_2 = \{f(1) = 1, f(2) = 1, g = u\}$ is an interpretation such that $J_2 \prec^{(f,g)} I$ and J_2 satisfies $gr_I[F]^{\underline{I}}$.

Advantages of these Coincidences

Establishing classes of formulas on which the semantics coincide provides two advantages:

- Implementations of one semantics can be used for the computation of the other semantics for any class of formula on which they coincide.
- Theoretical results that hold for one semantics hold for the other semantics for any class of formula on which they coincide.

Unfolding

The process of *unfolding* F *w.r.t.* $\mathbf{c}$, denoted by $UF_{\mathbf{c}}(F)$, is a process to transform a formula so that functions in $\mathbf{c}$ are not nested. Note that $UF_{\mathbf{c}}(F)$ is $\mathbf{c}$-plain.

- $UF_{(f,g)}(f = g)$ is $\exists xy(x = y \wedge f = x \wedge g = y)$.
- $UF_{(p,f,g)}(p(f(g)))$ is $\exists xy(p(x) \wedge f(y) = x \wedge g = y)$.

Unfolding

Unfolding preserves the partial equilibrium models. This theorem generalizes the unfolding result in [Cabalar 2011].

Theorem (7)

For any sentence F, any list $\mathbf{c}$ *of intensional constants, and any partial interpretation I, I is a partial equilibrium model of F iff I is a partial equilibrium model of $UF_{\mathbf{c}}(F)$ iff I is a stable model of $UF_{\mathbf{c}}(F)$ according to [Bartholomew and Lee 2012].*

Unfolding does not preserve the stable models of a formula in general. However, it follows that if F is a tight head-$\mathbf{c}$-plain formula, then F and $UF_{\mathbf{c}}(F)$ have the same stable models:

- If F is a tight head-$\mathbf{c}$-plain formula, then a total interpretation I is a stable model of F iff I is a partial equilibrium model of F.
- By this theorem, $UF_{\mathbf{c}}(F)$ and F have the same partial equilibrium models.
- Since $UF_{\mathbf{c}}(F)$ is $\mathbf{c}$-plain, a total interpretation I is a stable model of $UF_{\mathbf{c}}(F)$ iff I is a partial equilibrium model of $UF_{\mathbf{c}}(F)$.

Comparing the Cabalar Semantics and the Balduccini Semantics

Balducci Semantics [Balduccini 2012]

The Balduccini Semantics is closely related to the reduct reformulation of the Cabalar semantics. Rather than considering interpretations, the Balduccini semantics considers consistent sets of seed literals of the form

- $f(\mathbf{t}) = c$ where f is an intensional function constant and c and each $t \in \mathbf{t}$ is a non-intensional object constant.
- $p(\mathbf{t})$ where each $t \in \mathbf{t}$ is a non-intensional object constant.

Sets of seed literals can be identified with partial interpretations:

$$\begin{aligned} S &= \{f(1) = 1, & & p(1)\} \\ I &= \{f(1) = 1, & f(2) = u, & p(1)\} \end{aligned}$$

The notion of satisfaction of a consistent set S of seed literals for a formula F is similar to partial satisfaction when we identify S with a partial interpretation.

Balduccini Semantics [Balduccini 2012]

An ASP{f} program is comprised of a finite set of rules of the form

$$h \leftarrow l_1, \ldots, l_m, not\ l_{m+1}, \ldots, not\ l_n \tag{1}$$

where h is a seed literal or $\bot$ and each l_i is a variable-free atomic formula. The reduct of ASP{f} program Π relative to a consistent set I of seed literals is denoted $\Pi^{\underline{I}}$ and is defined as

$$\Pi^{\underline{I}} = \{h \leftarrow l_1, \ldots, l_m \mid (1) \in \Pi \text{ and } I \models \neg l_{m+1} \wedge \cdots \wedge \neg l_n\}$$

I is called a Balduccini answer set of Π if

- $I \models_{\overline{\mathrm{b}}} \Pi^{\underline{I}}$, and,
- for every proper subset J of I, we have $J \not\models_{\overline{\mathrm{b}}} \Pi^{\underline{I}}$.

Comparison between Balduccini and Cabalar Semantics

Theorem (9)

For any ASP$\{f\}$ program Π with intensional constants $\mathbf{c}$ and any consistent set I of seed literals, if Π contains no strong negation, then I is a Balduccini answer set of Π iff I is a partial equilibrium model of Π.

This theorem is also extended to consider ASP{f} programs with strong negation.

Representing Strong Negation in the Cabalar Semantics

The Cabalar semantics does not explicitly support strong negation. Instead, Cabalar defines a construct $\#$ where $f \# g$ is an abbreviation for $(f = f) \wedge (g = g) \wedge \neg(f = g)$.

- $I_1 = \{f = 1, g = 2\}$ is a model of $f \# g$.
- $I_2 = \{f = u, g = u\}$ and $I_3 = \{f = u, g = 1\}$ are not models of $f \# g$.

We can identify $\sim(f = g)$ in the sense of Balduccini with $(f = f) \wedge (g = g) \wedge \neg(f = g)$ in the sense of Cabalar.

Conclusion

- We provided reformulations of the semantics

	Bartholomew-Lee Semantics	Cabalar Semantics	Balduccini Semantics
Second-order Logic	[Bartholomew and Lee 2012]	This paper	
Variant of HT Logic	This paper, [del Cerro et al. 2013]	[Cabalar 2011]	
Reduct	This paper	This paper	[Balduccini 2012]

- The Bartholomew-Lee and Cabalar semantics coincide on **c**-plain formulas and tight head-**c**-plain formulas.
- The Balduccini semantics is a special case of the Cabalar Semantics.
- Existing implementations for one semantics can be used to compute another semantics for either restricted class of formula, e.g. Cabalar semantics can be computed by SMT solvers.
- Theoretical developments for one semantics can now more easily be adapted to another semantics for these restricted classes of formula, e.g. unfolding.

Questions