

Fuzzy Answer Sets Approximations

Mario Alviano¹ and Rafael Peñaloza²

¹Department of Mathematics and Computer Science
University of Calabria

²Center for Advancing Electronics Dresden
Dresden University of Technology

ICLP 2013

Istanbul, 27 August 2013

- 1 Introduction
- 2 Syntax and Semantics
- 3 Approximation Operators
 - Immediate Consequence Operator
 - Minimal Satisfiability
 - Well-founded Operator
- 4 Implementation and Experiment
- 5 Conclusion

Answer Set Programming (ASP)

- Overcomes a weakness of classical logic for KR: monotonicity
- Naturally handles reasoning by defaults, abductive reasoning, belief revisions, ...

ASP makes logic closer to the real world

However...

- Everything is either true or false in ASP
- ASP is based on precise information

Can we always make these assumptions?

Answer Set Programming (ASP)

- Overcomes a weakness of classical logic for KR: monotonicity
- Naturally handles reasoning by defaults, abductive reasoning, belief revisions, ...

ASP makes logic closer to the real world

However...

- Everything is either true or false in ASP
- ASP is based on precise information

Can we always make these assumptions?

Barber of Seville paradox

In the small town of Seville, all and only those men who do not shave themselves are shaved by the barber (who is a man).
Who shaves the barber?

$$\begin{aligned} \textit{shaves}(\textit{barber}, X) &\leftarrow \textit{not shaves}(X, X) \\ \textit{shaves}(X, X) &\leftarrow \textit{not shaves}(\textit{barber}, X) \end{aligned}$$

- Classical set theory can neither prove nor disprove that the barber shaves himself
- An odd loop makes the program incoherent for ASP
- $\textit{shaves}(\textit{barber}, \textit{barber})$ is **undefined** according to the well-founded semantics

Barber of Seville paradox

In the small town of Seville, all and only those men who do not shave themselves are shaved by the barber (who is a man).
Who shaves the barber?

$shaves(barber, barber) \leftarrow not\ shaves(barber, barber)$
 $shaves(barber, barber) \leftarrow not\ shaves(barber, barber)$

- Classical set theory can neither prove nor disprove that the barber shaves himself
- An odd loop makes the program incoherent for ASP
- $shaves(barber, barber)$ is **undefined** according to the well-founded semantics

B

How many truth degrees?

- Let's associate false statements with white, and true statements with black
- Undefined statements can be associated to gray
- Can we model statements which are more likely to be false or to be true?
- ... and we can go on!

A**C****B**

How many truth degrees?

- Let's associate false statements with white, and true statements with black
- Undefined statements can be associated to gray
- Can we model statements which are more likely to be false or to be true?
- ... and we can go on!

A**C****E****B**

How many truth degrees?

- Let's associate false statements with white, and true statements with black
- Undefined statements can be associated to gray
- Can we model statements which are more likely to be false or to be true?
- ... and we can go on!

A D C H E B

How many truth degrees?

- Let's associate false statements with white, and true statements with black
- Undefined statements can be associated to gray
- Can we model statements which are more likely to be false or to be true?
- ... and we can go on!

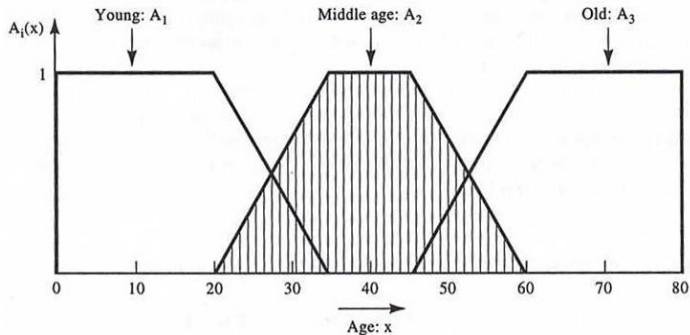
A F D G C H E I B

How many truth degrees?

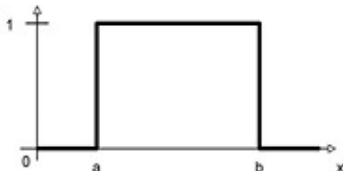
- Let's associate false statements with white, and true statements with black
- Undefined statements can be associated to gray
- Can we model statements which are more likely to be false or to be true?
- ... and we can go on!

Only precise information?

- When a person stops to be young and becomes old?



- Compare it with the classical, crisp approach



How to Handle Vagueness?

Fuzzy logic interprets propositions with a truth degree in $[0,1]$

Classical (crisp) set vs. fuzzy set



Fuzzification



Fuzzy inference engine



Defuzzification

This work deals with
the engine part

Possible Application

Firma di Insegnante		Firma		Data: 18/03/2013	
Studente	Firma	Studente	Firma	Studente	Firma
1. ADRIANO		14. CALABRO		28. MARCHIO	
2. AMORIO		15. CALABRO		29. MARCHIO	
3. ANTONI		16. CALABRO		30. MARCHIO	
4. ANTONI		17. CALABRO		31. MARCHIO	
5. ANTONI		18. CALABRO		32. MARCHIO	
6. ANTONI		19. CALABRO		33. MARCHIO	
7. ANTONI		20. CALABRO		34. MARCHIO	
8. ANTONI		21. CALABRO		35. MARCHIO	
9. ANTONI		22. CALABRO		36. MARCHIO	
10. ANTONI		23. CALABRO		37. MARCHIO	
11. ANTONI		24. CALABRO		38. MARCHIO	
12. ANTONI		25. CALABRO		39. MARCHIO	
13. ANTONI		26. CALABRO		40. MARCHIO	
14. ANTONI		27. CALABRO		41. MARCHIO	
15. ANTONI		28. CALABRO		42. MARCHIO	
16. ANTONI		29. CALABRO		43. MARCHIO	
17. ANTONI		30. CALABRO		44. MARCHIO	
18. ANTONI		31. CALABRO		45. MARCHIO	
19. ANTONI		32. CALABRO		46. MARCHIO	
20. ANTONI		33. CALABRO		47. MARCHIO	
21. ANTONI		34. CALABRO		48. MARCHIO	
22. ANTONI		35. CALABRO		49. MARCHIO	
23. ANTONI		36. CALABRO		50. MARCHIO	
24. ANTONI		37. CALABRO		51. MARCHIO	
25. ANTONI		38. CALABRO		52. MARCHIO	
26. ANTONI		39. CALABRO		53. MARCHIO	
27. ANTONI		40. CALABRO		54. MARCHIO	
28. ANTONI		41. CALABRO		55. MARCHIO	
29. ANTONI		42. CALABRO		56. MARCHIO	
30. ANTONI		43. CALABRO		57. MARCHIO	
31. ANTONI		44. CALABRO		58. MARCHIO	
32. ANTONI		45. CALABRO		59. MARCHIO	
33. ANTONI		46. CALABRO		60. MARCHIO	
34. ANTONI		47. CALABRO		61. MARCHIO	
35. ANTONI		48. CALABRO		62. MARCHIO	
36. ANTONI		49. CALABRO		63. MARCHIO	
37. ANTONI		50. CALABRO		64. MARCHIO	
38. ANTONI		51. CALABRO		65. MARCHIO	
39. ANTONI		52. CALABRO		66. MARCHIO	
40. ANTONI		53. CALABRO		67. MARCHIO	
41. ANTONI		54. CALABRO		68. MARCHIO	
42. ANTONI		55. CALABRO		69. MARCHIO	
43. ANTONI		56. CALABRO		70. MARCHIO	
44. ANTONI		57. CALABRO		71. MARCHIO	
45. ANTONI		58. CALABRO		72. MARCHIO	
46. ANTONI		59. CALABRO		73. MARCHIO	
47. ANTONI		60. CALABRO		74. MARCHIO	
48. ANTONI		61. CALABRO		75. MARCHIO	
49. ANTONI		62. CALABRO		76. MARCHIO	
50. ANTONI		63. CALABRO		77. MARCHIO	
51. ANTONI		64. CALABRO		78. MARCHIO	
52. ANTONI		65. CALABRO		79. MARCHIO	
53. ANTONI		66. CALABRO		80. MARCHIO	
54. ANTONI		67. CALABRO		81. MARCHIO	
55. ANTONI		68. CALABRO		82. MARCHIO	
56. ANTONI		69. CALABRO		83. MARCHIO	
57. ANTONI		70. CALABRO		84. MARCHIO	
58. ANTONI		71. CALABRO		85. MARCHIO	
59. ANTONI		72. CALABRO		86. MARCHIO	
60. ANTONI		73. CALABRO		87. MARCHIO	
61. ANTONI		74. CALABRO		88. MARCHIO	
62. ANTONI		75. CALABRO		89. MARCHIO	
63. ANTONI		76. CALABRO		90. MARCHIO	
64. ANTONI		77. CALABRO		91. MARCHIO	
65. ANTONI		78. CALABRO		92. MARCHIO	
66. ANTONI		79. CALABRO		93. MARCHIO	
67. ANTONI		80. CALABRO		94. MARCHIO	
68. ANTONI		81. CALABRO		95. MARCHIO	
69. ANTONI		82. CALABRO		96. MARCHIO	
70. ANTONI		83. CALABRO		97. MARCHIO	
71. ANTONI		84. CALABRO		98. MARCHIO	
72. ANTONI		85. CALABRO		99. MARCHIO	
73. ANTONI		86. CALABRO		100. MARCHIO	

42% (2)	Bonito Tomerino
50% (8)	Pateras Carnis
48% (8)	San Vito
15% (8)	San Vito
0% (0)	

Possible solution

$top(X, high) \leftarrow top(X, medium),$
 not $bottom(X - 1, high).$
 $bottom(X, high) \leftarrow bottom(X, medium),$
 not $top(X + 1, high).$
 $sign(X, l) \leftarrow top(X, l).$
 $sign(X, l) \leftarrow bottom(X, l).$

Possible Application

Firma di Insegnante		Firma		Data: 18/03/2013	
Studente	Firma	Studente	Firma	Studente	Firma
1. ADRIANO		14. CARICATO		28. MARCO	
2. ADRIANO		15. CARICATO		29. MARCO	
3. ADRIANO		16. CARICATO		30. MARCO	
4. ADRIANO		17. CARICATO		31. MARCO	
5. ADRIANO		18. CARICATO		32. MARCO	
6. ADRIANO		19. CARICATO		33. MARCO	
7. ADRIANO		20. CARICATO		34. MARCO	
8. ADRIANO		21. CARICATO		35. MARCO	
9. ADRIANO		22. CARICATO		36. MARCO	
10. ADRIANO		23. CARICATO		37. MARCO	
11. ADRIANO		24. CARICATO		38. MARCO	
12. ADRIANO		25. CARICATO		39. MARCO	
13. ADRIANO		26. CARICATO		40. MARCO	
14. ADRIANO		27. CARICATO		41. MARCO	
15. ADRIANO		28. CARICATO		42. MARCO	
16. ADRIANO		29. CARICATO		43. MARCO	
17. ADRIANO		30. CARICATO		44. MARCO	
18. ADRIANO		31. CARICATO		45. MARCO	
19. ADRIANO		32. CARICATO		46. MARCO	
20. ADRIANO		33. CARICATO		47. MARCO	
21. ADRIANO		34. CARICATO		48. MARCO	
22. ADRIANO		35. CARICATO		49. MARCO	
23. ADRIANO		36. CARICATO		50. MARCO	
24. ADRIANO		37. CARICATO		51. MARCO	
25. ADRIANO		38. CARICATO		52. MARCO	
26. ADRIANO		39. CARICATO		53. MARCO	
27. ADRIANO		40. CARICATO		54. MARCO	
28. ADRIANO		41. CARICATO		55. MARCO	
29. ADRIANO		42. CARICATO		56. MARCO	
30. ADRIANO		43. CARICATO		57. MARCO	
31. ADRIANO		44. CARICATO		58. MARCO	
32. ADRIANO		45. CARICATO		59. MARCO	
33. ADRIANO		46. CARICATO		60. MARCO	
34. ADRIANO		47. CARICATO		61. MARCO	
35. ADRIANO		48. CARICATO		62. MARCO	
36. ADRIANO		49. CARICATO		63. MARCO	
37. ADRIANO		50. CARICATO		64. MARCO	
38. ADRIANO		51. CARICATO		65. MARCO	
39. ADRIANO		52. CARICATO		66. MARCO	
40. ADRIANO		53. CARICATO		67. MARCO	
41. ADRIANO		54. CARICATO		68. MARCO	
42. ADRIANO		55. CARICATO		69. MARCO	
43. ADRIANO		56. CARICATO		70. MARCO	
44. ADRIANO		57. CARICATO		71. MARCO	
45. ADRIANO		58. CARICATO		72. MARCO	
46. ADRIANO		59. CARICATO		73. MARCO	
47. ADRIANO		60. CARICATO		74. MARCO	
48. ADRIANO		61. CARICATO		75. MARCO	
49. ADRIANO		62. CARICATO		76. MARCO	
50. ADRIANO		63. CARICATO		77. MARCO	
51. ADRIANO		64. CARICATO		78. MARCO	
52. ADRIANO		65. CARICATO		79. MARCO	
53. ADRIANO		66. CARICATO		80. MARCO	
54. ADRIANO		67. CARICATO		81. MARCO	
55. ADRIANO		68. CARICATO		82. MARCO	
56. ADRIANO		69. CARICATO		83. MARCO	
57. ADRIANO		70. CARICATO		84. MARCO	
58. ADRIANO		71. CARICATO		85. MARCO	
59. ADRIANO		72. CARICATO		86. MARCO	
60. ADRIANO		73. CARICATO		87. MARCO	
61. ADRIANO		74. CARICATO		88. MARCO	
62. ADRIANO		75. CARICATO		89. MARCO	
63. ADRIANO		76. CARICATO		90. MARCO	
64. ADRIANO		77. CARICATO		91. MARCO	
65. ADRIANO		78. CARICATO		92. MARCO	
66. ADRIANO		79. CARICATO		93. MARCO	
67. ADRIANO		80. CARICATO		94. MARCO	
68. ADRIANO		81. CARICATO		95. MARCO	
69. ADRIANO		82. CARICATO		96. MARCO	
70. ADRIANO		83. CARICATO		97. MARCO	
71. ADRIANO		84. CARICATO		98. MARCO	
72. ADRIANO		85. CARICATO		99. MARCO	
73. ADRIANO		86. CARICATO		100. MARCO	

42%	Bonito Tommaso
50%	Patera Carnis
48%	Tommaso Voe
15%	
0% (0)	

Possible solution

$top(X, high) \leftarrow top(X, medium),$
 not $bottom(X - 1, high).$
 $bottom(X, high) \leftarrow bottom(X, medium),$
 not $top(X + 1, high).$
 $sign(X, l) \leftarrow top(X, l).$
 $sign(X, l) \leftarrow bottom(X, l).$

Fuzzy inference engine

- Minimal satisfiability operator
- Fuzzy unfounded sets
- Implementation
 - Grounder (gringo)
 - Solver: fuzzy answer sets (approximation) and well-founded model
- Experiment

- 1 Introduction
- 2 Syntax and Semantics
- 3 Approximation Operators
 - Immediate Consequence Operator
 - Minimal Satisfiability
 - Well-founded Operator
- 4 Implementation and Experiment
- 5 Conclusion

Normal FASP program: set of rules of the form

$$a \leftarrow b_1 \otimes \cdots \otimes b_m \otimes \text{not } b_{m+1} \otimes \cdots \otimes \text{not } b_n$$

Fuzzy atoms are either propositional atoms in a fixed set \mathcal{B} or numeric constants: $a, b, \text{speed}, \overline{0}, \overline{1}, \overline{0.5}, \overline{1/3}, \dots$

Example

$$a \leftarrow \text{not } b$$

$$b \leftarrow a \otimes \overline{0.8}$$

$\otimes : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a fixed **t-norm**, e.g.

- Gödel: $x \otimes y = \min\{x, y\}$
- Product: $x \otimes y = x \cdot y$
- Łukasiewicz: $x \otimes y = \max\{x + y - 1, 0\}$

Interpretation: $I : \mathcal{B} \rightarrow [0, 1]$

- Numeric constant: $I(\bar{c}) = c$
- Negative literal: $I(\text{not } a) = 1 - I(a)$
- Conjunction: $I(\bigotimes_{i=1}^k l_i) = \bigotimes_{i=1}^k I(l_i)$

Model: $I \models r$ if $I(H(r)) \geq I(B(r))$

Program Reduct: in P , replace each $\text{not } b$ by $\overline{I(\text{not } b)}$

Fuzzy Answer Set: $M \models P$ and there is no $I \subset M$ s.t. $I \models P^M$

Fuzzy Sets: Operations and Relations

$I \subseteq J$: $I(a) \leq J(a)$ for each $a \in \mathcal{B}$

$I \subset J$: $I \subseteq J$ and $I \neq J$

$I \cap J$: $a \mapsto \min\{I(a), J(a)\}$ for each $a \in \mathcal{B}$

$I \cup J$: $a \mapsto \max\{I(a), J(a)\}$ for each $a \in \mathcal{B}$

$I \setminus J$: $a \mapsto \max\{I(a) - J(a), 0\}$ for each $a \in \mathcal{B}$

- 1 Introduction
- 2 Syntax and Semantics
- 3 Approximation Operators**
 - Immediate Consequence Operator
 - Minimal Satisfiability
 - Well-founded Operator
- 4 Implementation and Experiment
- 5 Conclusion

Goal: Compute Approximations

- Start with the largest approximation (**0, 1**)
 - Lower bound: all propositions associated with 0
 - Upper bound: all propositions associated with 1
- Apply operators to restrict the approximation
 - Increase the lower bound
 - Decrease the upper bound
- All the answer sets must be between these bounds

Definition

For each $a \in \mathcal{B}$, define

$$T_P^U(L) : a \mapsto \max\{\langle L, U \rangle(r) \mid r \in P, H(r) = a\}$$

For $r : a \leftarrow b_1 \otimes \cdots \otimes b_m \otimes \text{not } b_{m+1} \otimes \cdots \otimes \text{not } b_n$

let $\langle L, U \rangle(r) := L(b_1 \otimes \cdots \otimes b_m) \otimes U(\text{not } b_{m+1} \otimes \cdots \otimes \text{not } b_n)$

Łukasiewicz t-norm: $x \otimes y = \max\{x + y - 1, 0\}$

$$a \leftarrow b \otimes \text{not } c \quad L = \{a \mapsto 0, b \mapsto 0.8, c \mapsto 0.3\}$$

$$U = \{a \mapsto 1, b \mapsto 1, c \mapsto 0.3\}$$

$$T_P^U(L) : a \mapsto L(b) \otimes U(\text{not } c) = \max\{0.8 + (1 - 0.3) - 1, 0\} = 0.5$$

Theorem

The fixpoint $T_P^U \uparrow \mathbf{0}$ is reached after a linear number of iterations, measured on the number of atoms appearing in P .

Definition

For each $a \in \mathcal{B}$, define

$$T_P^U(L) : a \mapsto \max\{\langle L, U \rangle(r) \mid r \in P, H(r) = a\}$$

For $r : a \leftarrow b_1 \otimes \cdots \otimes b_m \otimes \text{not } b_{m+1} \otimes \cdots \otimes \text{not } b_n$

let $\langle L, U \rangle(r) := L(b_1 \otimes \cdots \otimes b_m) \otimes U(\text{not } b_{m+1} \otimes \cdots \otimes \text{not } b_n)$

Łukasiewicz t-norm: $x \otimes y = \max\{x + y - 1, 0\}$

$$a \leftarrow b \otimes \text{not } c \quad L = \{a \mapsto 0, b \mapsto 0.8, c \mapsto 0.3\}$$

$$U = \{a \mapsto 1, b \mapsto 1, c \mapsto 0.3\}$$

$$T_P^U(L) : a \mapsto L(b) \otimes U(\text{not } c) = \max\{0.8 + (1 - 0.3) - 1, 0\} = 0.5$$

Theorem

The fixpoint $T_P^U \uparrow \mathbf{0}$ is reached after a linear number of iterations, measured on the number of atoms appearing in P .

Definition

For each $a \in \mathcal{B}$, define

$$S_P^U(L) : a \mapsto \inf\{I(a) \mid I \models P \wedge L \subseteq I \subseteq U\}$$

Theorem

If $M \models P$ and $L \subseteq M \subseteq U$ then $S_P^U(L) \subseteq M \subseteq U$.

How to compute this operator?

For the Łukasiewicz t-norm, rewrite each

$$a \leftarrow b_1 \otimes \cdots \otimes b_m \otimes \text{not } b_{m+1} \otimes \cdots \otimes \text{not } b_n$$

into

$$a \geq b_1 + \cdots + b_m - b_{m+1} - \cdots - b_n + 1 - m$$

and solve a system of linear inequalities.

Definition

For each $a \in \mathcal{B}$, define

$$S_P^U(L) : a \mapsto \inf\{I(a) \mid I \models P \wedge L \subseteq I \subseteq U\}$$

Theorem

If $M \models P$ and $L \subseteq M \subseteq U$ then $S_P^U(L) \subseteq M \subseteq U$.

How to compute this operator?

For the Łukasiewicz t-norm, rewrite each

$$a \leftarrow b_1 \otimes \cdots \otimes b_m \otimes \text{not } b_{m+1} \otimes \cdots \otimes \text{not } b_n$$

into

$$a \geq b_1 + \cdots + b_m - b_{m+1} - \cdots - b_n + 1 - m$$

and solve a system of linear inequalities.

Example

 $a \leftarrow \text{not } b$ $b \leftarrow a \otimes 0.8$ min a s.t. $a \geq 1 - b$ $b \geq a + 0.8 - 1$ $0 \leq a, b \leq 1$ $S_P^1(\mathbf{0}) : a \mapsto 0.6$ min b s.t. $a \geq 1 - b$ $b \geq a + 0.8 - 1$ $0 \leq a, b \leq 1$ $S_P^1(\mathbf{0}) : b \mapsto 0.4$

Theorem

The minimal satisfiability operator is computable in polynomial time for programs over the Łukasiewicz t-norm.

Definition

A fuzzy unfounded set X for P w.r.t. (L, U) satisfies

$$[U \cap (\mathbf{1} \setminus X)](H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L \rangle(r)$$

for each $r \in P$ such that $X(H(r)) > 0$.

Fuzzy unfounded sets evidence lack of (acyclic) support.

Łukasiewicz t-norm: $x \otimes y = \max\{x + y - 1, 0\}$

$$a \leftarrow b \otimes b$$

$$b \leftarrow a$$

$$b \leftarrow \overline{0.9} \otimes \text{not } c$$

$$c \leftarrow \overline{0.9} \otimes \text{not } b$$

Unfounded sets for $(\mathbf{0}, \mathbf{1})$:

$$X_1 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0\}$$

$$X_2 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0.1\}$$

$$X_3 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0\}$$

$$X_4 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0.1\}$$

Definition

A fuzzy unfounded set X for P w.r.t. (L, U) satisfies

$$\langle U \cap (\mathbf{1} \setminus X) \rangle (H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L \rangle (r)$$

for each $r \in P$ such that $X(H(r)) > 0$.

Fuzzy unfounded sets evidence lack of (acyclic) support.

Lukasiewicz t-norm: $x \otimes y = \max\{x + y - 1, 0\}$

$$a \leftarrow b \otimes b$$

$$b \leftarrow a$$

$$b \leftarrow \overline{0.9} \otimes \text{not } c$$

$$c \leftarrow \overline{0.9} \otimes \text{not } b$$

Unfounded sets for $(\mathbf{0}, \mathbf{1})$:

$$X_1 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0\}$$

$$X_2 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0.1\}$$

$$X_3 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0\}$$

$$X_4 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0.1\}$$

Definition

A fuzzy unfounded set X for P w.r.t. (L, U) satisfies

$$[U \cap (\mathbf{1} \setminus X)](H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L \rangle(r)$$

for each $r \in P$ such that $X(H(r)) > 0$.

Fuzzy unfounded sets evidence lack of (acyclic) support.

Łukasiewicz t-norm: $x \otimes y = \max\{x + y - 1, 0\}$

$$a \leftarrow b \otimes b$$

$$b \leftarrow a$$

$$b \leftarrow \overline{0.9} \otimes \text{not } c$$

$$c \leftarrow \overline{0.9} \otimes \text{not } b$$

Unfounded sets for $(\mathbf{0}, \mathbf{1})$:

$$X_1 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0\}$$

$$X_2 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0.1\}$$

$$X_3 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0\}$$

$$X_4 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0.1\}$$

Definition

A fuzzy unfounded set X for P w.r.t. (L, U) satisfies

$$\langle U \cap (\mathbf{1} \setminus X) \rangle (H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L \rangle (r)$$

for each $r \in P$ such that $X(H(r)) > 0$.

Fuzzy unfounded sets evidence lack of (acyclic) support.

Łukasiewicz t-norm: $x \otimes y = \max\{x + y - 1, 0\}$

$$a \leftarrow b \otimes b$$

$$b \leftarrow a$$

$$b \leftarrow \overline{0.9} \otimes \text{not } c$$

$$c \leftarrow \overline{0.9} \otimes \text{not } b$$

Unfounded sets for $(\mathbf{0}, \mathbf{1})$:

$$X_1 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0\}$$

$$X_2 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0.1\}$$

$$X_3 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0\}$$

$$X_4 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0.1\}$$

Definition

A fuzzy unfounded set X for P w.r.t. (L, U) satisfies

$$\langle U \cap (\mathbf{1} \setminus X) \rangle (H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L \rangle (r)$$

for each $r \in P$ such that $X(H(r)) > 0$.

Fuzzy unfounded sets evidence lack of (acyclic) support.

Łukasiewicz t-norm: $x \otimes y = \max\{x + y - 1, 0\}$

$$a \leftarrow b \otimes b$$

$$b \leftarrow a$$

$$b \leftarrow \overline{0.9} \otimes \text{not } c$$

$$c \leftarrow \overline{0.9} \otimes \text{not } b$$

Unfounded sets for $(\mathbf{0}, \mathbf{1})$:

$$X_1 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0\}$$

$$X_2 = \{a \mapsto 0, b \mapsto 0, c \mapsto 0.1\}$$

$$X_3 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0\}$$

$$X_4 = \{a \mapsto 0.2, b \mapsto 0.1, c \mapsto 0.1\}$$

Lemma

The union of two (fuzzy) unfounded sets is an unfounded set.

Let $GUS_P^{L,U}$ denote the greatest fuzzy unfounded set.

Theorem

M is a fuzzy answer set of a program P iff $GUS_P^{M,M} = \mathbf{1} \setminus M$.

Theorem

For programs without numeric constants and crisp sets, fuzzy unfounded sets are the unfounded sets by Van Gelder et al. [1991].

Lemma

The union of two (fuzzy) unfounded sets is an unfounded set.

Let $GUS_P^{L,U}$ denote the greatest fuzzy unfounded set.

Theorem

M is a fuzzy answer set of a program P iff $GUS_P^{M,M} = \mathbf{1} \setminus M$.

Theorem

For programs without numeric constants and crisp sets, fuzzy unfounded sets are the unfounded sets by Van Gelder et al. [1991].

$$W_P(L, U) := \left(T_P^U(L), \mathbf{1} \setminus R_P^{L,U} \downarrow \mathbf{1} \right)$$

where for every $a \in \mathcal{B}$, define

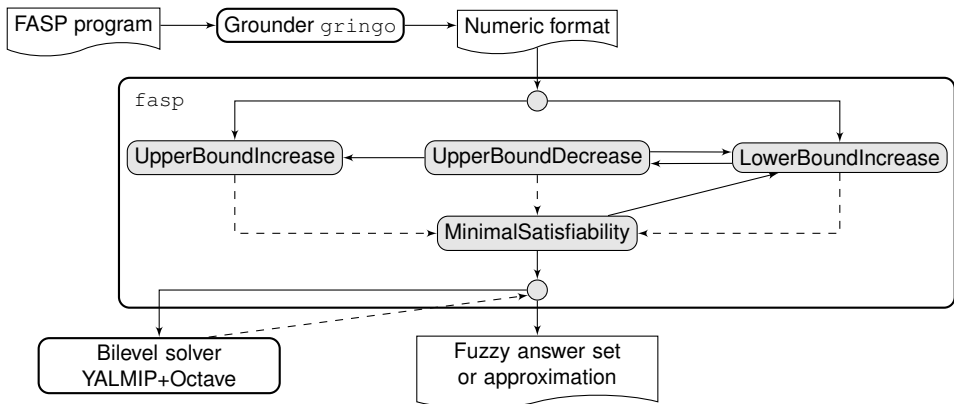
$$R_P^{L,U}(X) : a \mapsto \min\{X(a), \\ 1 - \max\{\langle U \cap (\mathbf{1} \setminus X), L \rangle(r) \mid r \in P, H(r) = a\}\}$$

Theorem

The fixpoint of W_P gives the well-founded semantics by Damásio and Pereira [2001].

- 1 Introduction
- 2 Syntax and Semantics
- 3 Approximation Operators
 - Immediate Consequence Operator
 - Minimal Satisfiability
 - Well-founded Operator
- 4 Implementation and Experiment
- 5 Conclusion

Implementation



<https://github.com/alviano/fasp.git>

FASP program

Grounder *gringo*

Numeric format

Example

$a \leftarrow \text{not } b$

$b \leftarrow a \otimes \overline{0.8}$

$\min a - a' + b - b'$

s.t. $0 \leq a, b \leq 1$

$\min a + b$

s.t. $a' \geq 1 - b$

$b' \geq a' + 0.8 - 1$

$0 \leq a', b' \leq 1$

LowerBoundIncrease

Bilevel solver
YALMIP+Octave

Fuzzy answer set
or approximation

<https://github.com/alviano/fasp.git>

	Tested instances	Timeouts		Average execution time*		Average perc. gain*
		Unopt.	Opt.	Unopt.	Optimized	
Graph Coloring	60	34	0	247.44	34.45 (2.68)	76.43%
Hamiltonian Path	40	33	9	120.51	6.41 (0.02)	81.49%
Stratified	90	10	0	190.07	1.80 (0.02)	96.71%
Odd Cycle	90	33	0	186.94	1.95 (0.03)	97.18%

* Computed on the instances solved by both the approaches.

- 1 Introduction
- 2 Syntax and Semantics
- 3 Approximation Operators
 - Immediate Consequence Operator
 - Minimal Satisfiability
 - Well-founded Operator
- 4 Implementation and Experiment
- 5 Conclusion

- Approximations provide a sensitive performance gain
- Still the bilevel approach seems to be too slow

Future Work

- More complex structures in the body (disjunction)
 - Compile into uninterpreted function symbols in `gringo`
 - Use a pre-processor in `fasp`
- How to deal with a choice operator?
 - Completion approach by Janssen et al. [2012]
 - Conflict analysis and learning?

Thank you!

- Approximations provide a sensitive performance gain
- Still the bilevel approach seems to be too slow

Future Work

- More complex structures in the body (disjunction)
 - Compile into uninterpreted function symbols in `gringo`
 - Use a pre-processor in `fasp`
- How to deal with a choice operator?
 - Completion approach by Janssen et al. [2012]
 - Conflict analysis and learning?

Thank you!

- Carlos Viegas Damásio and Luís Moniz Pereira. Antitonic logic programs. In Proceedings of the 6th International Conference on Logic Programming and Nonmonotonic Reasoning, LPNMR'01, pages 379–392, London, UK, UK, 2001. Springer-Verlag. ISBN 3-540-42593-4.
- Jeroen Janssen, Dirk Vermeir, Steven Schockaert, and Martine De Cock. Reducing fuzzy answer set programming to model finding in fuzzy logics. Theory and Practice of Logic Programming, 12(6):811–842, 2012.
- Allen Van Gelder, Kenneth A. Ross, and John S. Schlipf. The Well-Founded Semantics for General Logic Programs. Journal of the ACM, 38(3):620–650, 1991.